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ON SIMPLE ALTERNATIVE RINGS

A. A. ALBERT

1. Introduction. The only known simple alternative rings which are not associative are the *Cayley algebras*. Every such algebra has a scalar extension which is isomorphic over its center F to the algebra $\mathsf{C} = e_{11}\mathsf{F} + e_{00}\mathsf{F} + \mathsf{C}_{10} + \mathsf{C}_{01}$, where $\mathsf{C}_{ij} = e_{ij}\mathsf{F} + f_{ij}\mathsf{F} + g_{ij}\mathsf{F}$ $(i, j = 0, 1;\ i \neq j)$. The elements e_{11} and e_{00} are orthogonal idempotents and $e_{ii}x_{ij} = x_{ij}e_{jj} = x_{ij}$, $e_{jj}x_{ij} = x_{ij}e_{ii} = 0$, $x_{ij}^2 = 0$ for every x_{ij} of C_{ij}. The multiplication table of C is then completed by the relations[1]

$$f_{10}g_{10} = e_{01},\ g_{10}e_{10} = f_{01},\ e_{10}f_{10} = g_{01}, \tag{1}$$
$$g_{01}f_{01} = e_{10},\ e_{01}g_{01} = f_{10},\ f_{01}e_{01} = g_{10}, \tag{2}$$
$$e_{ij}e_{ji} = f_{ij}f_{ji} = g_{ij}e_{ji} = e_{ii}, \tag{3}$$
$$e_{ij}f_{ji} = e_{ij}g_{ji} = f_{ij}e_{ji} = f_{ij}g_{ji} = g_{ij}e_{ji} = g_{ij}f_{ji} = 0. \tag{4}$$

R. H. Bruck and E. Kleinfeld have recently shown[2] that *every alternative division ring of characteristic not two is either associative or a Cayley algebra.* Their methods do not seem to be readily applicable to the simple case but we shall use the machinery of idempotents to prove the following result.

THEOREM. *Every simple alternative ring which contains an idempotent not its unity quantity is either associative or is the Cayley algebra* C.

2. Elementary properties. Our results are based on properties which were given by Zorn.[1] He assumed that the characteristic was not 2 or 3 and did not give complete details of his computations. As we shall make no assumption about the characteristic of our rings it will be necessary for us to re-derive the properties of Zorn and so make our exposition quite self-contained.

We first note that an alternative ring C is a mathematical system having the usual properties of associative rings except that the associative law for products is replaced by the identities $x(xy) = (xx)y$, $(yx)x = y(xx)$. It is easy to see that the *associator*

$$(x, y, z) = (xy)z - x(yz)$$

is an alternating function of its arguments x, y, z, a result which implies that

$$z(xy + yx) = (zx)y + (zy)x,\ (xy + yx)z = x(yz) + y(xz),$$
$$z(xy) + y(xz) = (zx)y + (yx)z, \tag{5}$$

for every x, y, z of C. We shall assume henceforth that C *contains an idempotent* u *not the unity quantity* of C.

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[1] The multiplication table of a Cayley algebra was given in this form by M. Zorn, *Theorie der alternativen Ringe*, Abh. Math. Sem. Hamburgischen Univ., vol. 8 (1930), 123–147.

[2] *The structure of alternative division rings*, Proc. Amer. Math. Soc., vol. 2 (1951), 878-890.

The ring C may be expressed as the module direct sum $\mathsf{C}=\mathsf{C}_{11}+\mathsf{C}_{10}+\mathsf{C}_{01}+\mathsf{C}_{00}$ of its submodules C_{ij} where C_{ij} consists of all x_{ij} of C such that $ux_{ij} = ix_{ij}$, $x_{ij}u = jx_{ij}$ $(i, j = 0, 1)$. Indeed if $x = x_{11} + x_{10} + x_{01} + x_{00}$ then $x_{11} = u(xu)$, $x_{10} = ux - u(xu)$, $x_{01} = xu - u(xu)$, $x_{00} = x - xu - ux - u(xu)$. This decomposition is precisely that of the associative case and needs no additional argument. However the multipicative properties of the modules C_{ij} need to be derived. We proceed as follows:

Let $ux = \lambda x$, $xu = \mu x$, $uy = \alpha y$, $yu = \beta y$. Then

$$\begin{aligned}(x, y, u) &= (xy)u - x(yu) = (xy)u - \beta xy = -(y, x, u) = y(xu) - (yx)u\\ &= \mu yx - (yx)u = -(x, u, y) = x(uy) - (xu)y = (\alpha - \mu)xy = (y, u, x)\\ &= (yu)x - y(ux) = (\beta - \lambda)yx = (u, x, y) = (ux)y - u(xy)\\ &= \lambda xy - u(xy) = -(u, y, x) = u(yx) - (uy)x = u(yx) - \alpha yx.\end{aligned}$$

We thus obtain the identities

$$(xy)u = (\alpha + \beta - \mu)xy,\quad u(xy) = (\lambda + \mu - \alpha)xy, \tag{6}$$

$$(yx)u = (\lambda + \mu - \beta)yx,\quad u(yx) = (\alpha + \beta - \lambda)yx, \tag{7}$$

$$(\alpha - \mu)xy = (\beta - \lambda)yx, \tag{8}$$

where (7) is obviously derivable from (6) by the interchange of x and y and the consequent interchanges of λ, μ with α, β. If $\lambda = \mu = \alpha = \beta = 1$ we have $(xy)u = u(xy) = xy$ and so C_{11} is a subring of C. Similarly the values $\lambda = \mu = \alpha = \beta = 0$ yield $u(xy) = (xy)u = 0$ and so C_{00} is a subring of C. We now put $\lambda = \mu = 1$ and $\alpha = \beta = 0$ to obtain $xy = yx$, $(xy)u = -xy$, $(yx)u = 2yx$, and so $(xy)u = 2xy$, $3xy = 0$. But $[(xy)u]u = -(xy)u = xy = (xy)u^2 = (xy)u = -xy$ and $2xy = 0$, $xy = 0$. This proves[3] that C_{11} *and and* C_{00} *are orthogonal subrings of* C.

We next put $\lambda = \mu = 1 = \alpha$ and $\beta = 0$. Then $(xy)u = 0$ and $u(xy) = xy$, $yx = 0$, and so $\mathsf{C}_{11}\mathsf{C}_{10} \subseteq \mathsf{C}_{10}$, $\mathsf{C}_{10}\mathsf{C}_{11} = 0$. By symmetry $\mathsf{C}_{01}\mathsf{C}_{11} \subseteq \mathsf{C}_{01}$, $\mathsf{C}_{11}\mathsf{C}_{01} = 0$. Similarly, the values $\lambda = \mu = \beta = 0$ and $\alpha = 1$ yield $xy = 0$, $u(yx) = yx$, $(yx)u = 0$, and so $\mathsf{C}_{00}\mathsf{C}_{10} = 0$, $\mathsf{C}_{10}\mathsf{C}_{00} \subseteq \mathsf{C}_{10}$, and $\mathsf{C}_{01}\mathsf{C}_{00} = 0$, $\mathsf{C}_{00}\mathsf{C}_{01} \subseteq \mathsf{C}_{01}$ by symmetry. The relations $\mathsf{C}_{10}\mathsf{C}_{01} \subseteq \mathsf{C}_{11}$, $\mathsf{C}_{01}\mathsf{C}_{10} \subseteq \mathsf{C}_{00}$ follow from (6) ,(7) by taking $\lambda = \beta = 1$, $\alpha = \mu = 0$.

The properties derived so far for the component modules C_{ij} are properties satisfied by all associative rings. In the associative case $\mathsf{C}_{10}{}^2 = \mathsf{C}_{01}{}^2 = 0$. However, this last result need not hold in the alternative case, and we now put $\alpha = \lambda = 1$, $\mu = \beta = 0$ and obtain $(xy)u = xy$, $u(xy) = 0$, $xy = -yx$. Thus we have the property

$$x_{10}y_{10} = -y_{10}x_{10} = z_{01}, \tag{9}$$

for every x_{10} and y_{10} of C_{10}, where z_{01} is in C_{01}. Similarly

$$x_{01}y_{01} = -y_{01}x_{01} = z_{10}. \tag{10}$$

[3]This seems to be one of the few places in our development where an assumption about the characteristic would make any difference.

Now $x_{10}^2 = (ux_{10})x_{10} = ux_{10}^2 = uz_{01} = 0$, and by symmetry we have the relation

$$x_{ij}^2 = 0 \qquad (i, j = 0, 1;\ i \neq j). \tag{11}$$

Zorn also gave the following result:

LEMMA 1. *Let x, y, z be elements of the component modules of C not all in the same subring C_{ii}. Then $(x, y, z) = 0$ except possibly when at least two of the elements are in the same module C_{ij} $(i \neq j)$.*

We also have the identities

$$z_{jj}(x_{ij}y_{ij}) = (x_{ij}z_{jj})y_{ij} = x_{ij}(y_{ij}z_{jj}), \tag{12}$$

$$(x_{ij}y_{ij})z_{ii} = (z_{ii}x_{ij})y_{ij} = x_{ij}(z_{ii}y_{ij}), \tag{13}$$

$$x_{ij}(y_{ji}z_{ii}) = z_{ii}(x_{ij}y_{ji}) = (z_{ii}x_{ij})y_{ji}, \tag{14}$$

$$x_{ij}(y_{ij}z_{ij}) = z_{ij}(x_{ij}y_{ij}) = y_{ij}(z_{ij}x_{ij}), \tag{15}$$

$$(x_{ij}y_{ij})z_{ij} = (z_{ij}x_{ij})y_{ij} = (y_{ij}z_{ij})x_{ij}. \tag{16}$$

We use (5) to write

$$z_{ii}(x_{ij}y_{ji}) - x_{ij}(y_{ji}z_{ii}) = z_{ii}(x_{ij}y_{ji}) + x_{ij}(z_{ii}y_{ji}) = (z_{ii}x_{ij} + x_{ij}z_{ii})y_{ji} = (z_{ii}x_{ij})y_{ji}$$

since $(x_{ij}z_{ii})y_{ji} = 0$. This proves (14). Also

$$(x_{ij}y_{ij})z_{ii} + (z_{ii}y_{ij})x_{ij} = x_{ij}(y_{ij}z_{ii}) + z_{ii}(y_{ij}x_{ij}) = 0$$

and so $(x_{ij}y_{ij})z_{ii} = -(z_{ii}y_{ij})x_{ij} = x_{ij}(z_{ii}y_{ij})$. Interchange x and y to obtain $(y_{ij}x_{ij})z_{ii} = -(z_{ii}x_{ij})y_{ij} = -(x_{ij}y_{ij})z_{ii}$ and we have proved (13). Formula (12) follows by symmetry. Now $(x_{ii}, y_{ii}, z_{jj}) = 0$ trivially,

$$(x_{ii}, y_{ii}, z_{ij}) = -(x_{ii}, z_{ij}, y_{ii}) = x_{ii}(z_{ij}y_{ii}) - (x_{ii}z_{ij})y_{ii} = 0,$$
$$(x_{ii}, y_{ij}, z_{jj}) = -(y_{ij}, x_{ii}, z_{jj}) = y_{ij}(x_{ii}z_{jj}) - (y_{ij}x_{ii})z_{jj} = 0,$$
$$(x_{ii}, y_{ij}, z_{ji}) = -(y_{ij}, x_{ii}, z_{ji}) = y_{ij}(x_{ii}z_{ji}) - (y_{ij}x_{ii})z_{ji} = 0.$$

The remaining properties of the associator follow by symmetry. Formula (15) states that the factors in $x_{ij}(y_{ij}z_{ij})$ may be permuted cyclically. To prove this result we use the final relation in (5) to write

$$z_{ij}(x_{ij}y_{ij}) + y_{ij}(x_{ij}z_{ij}) = (z_{ij}x_{ij})y_{ij} + (y_{ij}x_{ij})z_{ij}.$$

The left member is in $\mathsf{C}_{ij}\mathsf{C}_{ij}^2 \subseteq \mathsf{C}_{ij}\mathsf{C}_{ji} \subseteq \mathsf{C}_{ii}$ and the right member is in $\mathsf{C}_{ij}^2\mathsf{C}_{ij} \subseteq \mathsf{C}_{ji}\mathsf{C}_{ij} \subseteq \mathsf{C}_{jj}$. Since $i \neq j$ both members vanish and we have

$$-y_{ij}(x_{ij}z_{ij}) = y_{ij}(z_{ij}x_{ij}), \quad (z_{ij}x_{ij})y_{ij} = -(y_{ij}x_{ij})z_{ij} = (x_{ij}y_{ij})z_{ij}$$

from which we have both (15) and (16).

COROLLARY. *The ring C is associative if and only if both C_{11} and C_{00} are associative and $\mathsf{C}_{10}^2 = \mathsf{C}_{01}^2 = 0$.*

3. Construction of ideals. We first consider the product $p_{ii} = x_{ij}(y_{ji}z_{ii})$ which is an element of $\mathsf{C}_{ij}\mathsf{C}_{ij}^2 \subseteq \mathsf{C}_{ii}$ and let a_{ii} be any element of C_{ii}. Then $a_{ii}p_{ii} = (a_{ii}x_{ij})(y_{ij}z_{ij})$ by Lemma 1. But then (15) and (13) imply that

$$a_{ii}p_{ii} = z_{ij}[(a_{ii}x_{ij})y_{ij}] = z_{ij}[(x_{ij}y_{ij})a_{ii}] = [z_{ij}(x_{ij}y_{ij})]a_{ii} = p_{ii}a_{ii}$$

for every a_{ii} of C_{ii} and p_{ii} of $\mathsf{C}_{ij}\mathsf{C}_{ij}^2$, $\mathsf{C}_{ii}(\mathsf{C}_{ij}\mathsf{C}_{ij}^2) = (\mathsf{C}_{ij}\mathsf{C}_{ij}^2)\mathsf{C}_{ii} \subseteq \mathsf{C}_{ij}\mathsf{C}_{ij}^2$. If b_{ii} is in C_{ii} then

$$\begin{aligned} b_{ii}(a_{ii}p_{ii}) &= b_{ii}[(a_{ii}x_{ij})(y_{ij}z_{ij})] = [b_{ii}(a_{ii}x_{ij})](y_{ij}z_{ij}) \\ &= [(b_{ii}a_{ii})x_{ij}](y_{ij}z_{ij}) = (b_{ii}a_{ii})p_{ii}, \end{aligned}$$

and $\mathsf{C}_{ij}\mathsf{C}_{ij}^2$ is contained in the centre of C_{ii}. By symmetry we have the following result:

LEMMA 2. *The modules $\mathsf{C}_{ij}\mathsf{C}_{ij}^2$ and $\mathsf{C}_{ji}^2\mathsf{C}_{ji}$ are ideals of C_{ii} which are contained in the centre of C_{ii} $(i, j = 0, 1; i \neq j)$.*

We next prove the following result:

LEMMA 3. *Let B_i be an ideal of C_{ii}. Then*

$$\mathsf{D}_i = \mathsf{B}_i + \mathsf{B}_i\mathsf{C}_{ij} + \mathsf{C}_{ji}\mathsf{B}_i + (\mathsf{C}_{ji}\mathsf{B}_i)\mathsf{C}_{ij} \qquad (i, j = 0, 1; i \neq j) \tag{17}$$

is an ideal of C.

We have $(\mathsf{C}_{ji}\mathsf{B}_i)\mathsf{C}_{ij} = \mathsf{C}_{ji}(\mathsf{B}_i\mathsf{C}_{ij})$ by Lemma 1. We now compute

$$\mathsf{C}_{ii}\mathsf{D}_i = \mathsf{C}_{ii}\mathsf{B}_i + (\mathsf{C}_{ii}\mathsf{B}_i)\mathsf{C}_{ij} \subseteq \mathsf{D}_i, \quad \mathsf{D}_i\mathsf{C}_{ii} = \mathsf{B}_i\mathsf{C}_i + \mathsf{C}_{ji}(\mathsf{B}_i\mathsf{C}_{ii}) \subseteq \mathsf{D}_i,$$

$$\mathsf{C}_{ij}\mathsf{D}_i = \mathsf{C}_{ij}(\mathsf{C}_{ji}B_i) + \mathsf{C}_{ij}[(\mathsf{C}_{ji}\mathsf{B}_i)\mathsf{C}_{ij}] \subseteq \mathsf{C}_{ii}\mathsf{B}_i + [\mathsf{C}_{ij}(\mathsf{C}_{ji}\mathsf{B}_i)]\mathsf{C}_{ij} + (\mathsf{C}_{ji}\mathsf{B}_i)(\mathsf{C}_{ij}^2)$$

by (14). Then $\mathsf{C}_{ij}\mathsf{D}_i \subseteq \mathsf{B}_i + \mathsf{B}_i\mathsf{C}_{ij} + (\mathsf{C}_{ji}\mathsf{B}_i)\mathsf{C}_{ji} \subseteq \mathsf{B}_i + \mathsf{B}_i\mathsf{C}_{ij} + \mathsf{B}_i\mathsf{C}_{ji}^2 \subseteq \mathsf{D}_i$ since $\mathsf{C}_{ji}^2 \subseteq \mathsf{C}_{ij}$. Also

$$\mathsf{D}_i\mathsf{C}_{ij} = \mathsf{B}_i\mathsf{C}_{ij} + (\mathsf{B}_i\mathsf{C}_{ij})\mathsf{C}_{ij} + (\mathsf{C}_{ji}\mathsf{B}_i)\mathsf{C}_{ij} \subseteq \mathsf{B}_i\mathsf{C}_{ij} + C_{ij}^2\mathsf{B}_i + \mathsf{C}_{ji}(\mathsf{B}_i C_{ij}) \subseteq \mathsf{D}_i.$$

If we pass to a ring anti-isomorphic to C the module D_i is unchanged but C_{ji} is replaced by C_{ij}. Hence $\mathsf{C}_{ji}\mathsf{D}_i \subseteq \mathsf{D}_i$, $\mathsf{D}_i\mathsf{C}_{ji} \subseteq \mathsf{D}_i$. Finally

$$\begin{aligned} \mathsf{C}_{jj}\mathsf{D}_i &= \mathsf{C}_{jj}(\mathsf{C}_{ji}\mathsf{B}_i) + \mathsf{C}_{jj}[(\mathsf{C}_{ji}\mathsf{B}_i)\mathsf{C}_{ij}] \\ &= (\mathsf{C}_{jj}\mathsf{C}_{ji})\mathsf{B}_i + [\mathsf{C}_{jj}(\mathsf{C}_{ji}\mathsf{B}_i)]\mathsf{C}_{ij} \subseteq \mathsf{C}_{ji}\mathsf{B}_i + (\mathsf{C}_{ji}\mathsf{B}_i)\mathsf{C}_{ij} \subseteq \mathsf{D}_i, \end{aligned}$$

and $\mathsf{D}_i\mathsf{C}_{jj} = \mathsf{B}_i(\mathsf{C}_{ij}\mathsf{C}_{jj}) + (\mathsf{C}_{ji}\mathsf{B}_i)(\mathsf{C}_{ij}\mathsf{C}_{jj}) \subseteq \mathsf{D}_i$. This completes our proof.

LEMMA 4. *Let*

$$\mathsf{C}_{10}^2\mathsf{C}_{10} = \mathsf{C}_{10}\mathsf{C}_{10}^2 = \mathsf{C}_{01}^2\mathsf{C}_{01} = \mathsf{C}_{01}\mathsf{C}_{01}^2 = 0.$$

Then $\mathsf{G} = \mathsf{C}_{10}^2 + \mathsf{C}_{01}^2$ is a proper ideal of C.

We have

$$\mathsf{C}_{ii}\mathsf{G} = \mathsf{C}_{ii}\mathsf{C}_{ji}^2 = (\mathsf{C}_{ji}\mathsf{C}_{ii})\mathsf{C}_{ji} \subseteq \mathsf{C}_{ji}^2 \subseteq \mathsf{G}, \; \mathsf{G}\mathsf{C}_{ii} = \mathsf{C}_{ij}^2\mathsf{C}_{ii} = \mathsf{C}_{ij}(\mathsf{C}_{ii}\mathsf{C}_{ij}) \subseteq \mathsf{G}.$$

Also $\mathsf{C}_{ij}\mathsf{G} = \mathsf{C}_{ij}\mathsf{C}_{ji}^2 \subseteq \mathsf{C}_{ij}^2 \subseteq \mathsf{G}$, $\mathsf{G}\mathsf{C}_{ij} \subseteq \mathsf{C}_{ji}^2\mathsf{C}_{ij} \subseteq \mathsf{C}_{ij}^2 \subseteq \mathsf{G}$, as desired. Now $\mathsf{C}_{11} \neq 0$, C_{11} is not contained in G, and G is a proper ideal of C.

The constructions just given are sufficient for our needs and we proceed now to the simple case.

4. Simple rings. Lemma 1 implies that

$$\begin{aligned} x_{ii}[y_{ii}(z_{ij}w_{ji})] &= x_{ii}[(y_{ii}z_{ij})w_{ji}] = [x_{ii}(y_{ii}z_{ij})]w_{ji} \\ &= [(x_{ii}y_{ii})z_{ij}]w_{ji} = (x_{ii}y_{ii})(z_{ij}w_{ji}). \end{aligned}$$

Since $x_{ii}(z_{ij}w_{ji}) = (x_{ii}z_{ij})w_{ji}$ and $(z_{ij}w_{ji})x_{ii} = z_{ij}(w_{ji}x_{ii})$ we see that $\mathsf{C}_{ij}\mathsf{C}_{ji}$ is an associative ideal of C_{ii}. It follows immediately that $\mathsf{B} = \mathsf{C}_{10}\mathsf{C}_{01} + \mathsf{C}_{10} + \mathsf{C}_{01} + \mathsf{C}_{01}\mathsf{C}_{10}$ is an ideal of C. If C is simple and $\mathsf{B} = 0$ then C_{00} is a proper ideal of C, and $\mathsf{C} = \mathsf{C}_{11}$ has u as unity quantity contrary to hypothesis. Hence $\mathsf{B} = \mathsf{C}$, $\mathsf{C}_{ij}\mathsf{C}_{ji} = \mathsf{C}_{ii}$ is associative. If B_i were a non-zero proper ideal of C_{ii} the ideal D_i of Lemma 3 would be a non-zero proper ideal of C. Thus we have

LEMMA 5. *Let C be simple. Then C_{11} is a simple associative ring and C_{00} is either zero or a simple associative ring.*

When C is simple the set G of Lemma 4 cannot be a proper ideal of C. Hence C is either associative or $\mathsf{G} = \mathsf{C}_{10}^2 + \mathsf{C}_{01}^2 \neq 0$, one of the modules $\mathsf{C}_{10}\mathsf{C}_{10}^2$, $\mathsf{C}_{01}^2\mathsf{C}_{01}$, $\mathsf{C}_{10}^2\mathsf{C}_{10}$, $\mathsf{C}_{01}\mathsf{C}_{01}^2$ must not be zero. Let $\mathsf{C}_{ij}\mathsf{C}_{ij}^2 \neq 0$. By Lemma 2 we know that $\mathsf{B}_i = \mathsf{C}_{ij}\mathsf{C}_{ij}^2$ is a non-zero ideal of C_{ii}, by Lemma 5 that $\mathsf{B}_i = \mathsf{C}_{ii}$, C_{ii} coincides with its centre and must be a field. If $a_i = x_{ij}h_{ji} \neq 0$ where x_{ij} is in C_{ij} and y_{ji} is in C_{ij}^2 then

$$a_i^2 = a_i(x_{ij}y_{ji}) = (a_ix_{ij})y_{ji} = [x_{ij}(y_{ji}x_{ij})]y_{ji} \neq 0$$

and so $y_{ji}x_{ij} \neq 0$, $\mathsf{C}_{ij}^2\mathsf{C}_{ij} \neq 0$. The converse is obvious and so $\mathsf{C}_{ij}\mathsf{C}_{ij}^2 \neq 0$ if and only if $\mathsf{C}_{ij}^2\mathsf{C}_{ij} \neq 0$. It follows that both C_{11} and C_{00} are fields. Moreover, since we may pass to an anti-isomorphic ring if necessary, we may assume that $\mathsf{C}_{10}\mathsf{C}_{10}^2 \neq 0$. We now prove

LEMMA 6. *The rings C_{11} and C_{00} are isomorphic fields with unity quantities $u = e_{11}$ and e_{00} respectively, $e = e_{11} + e_{00}$ is the unity quantity of C, $e_{11} = e_{10}e_{01}$, $e_{00} = e_{01}e_{10}$ for quantities e_{ij} in C_{ij} such that $e_{01} = f_{10}g_{10}$ and f_{10}, g_{10} are in C_{10}.*

We select f_{11} and g_{10} so that $x_{10}e_{01} = a_1 \neq 0$ in the field C_{11}. Then a_1 has an inverse b_1 in C_{11} and $b_1(x_{10}e_{01}) = e_{11} = (b_1x_{10})e_{01} = e_{10}e_{01}$. Thus

$$e_{11}^2 = e_{11}(e_{10}e_{01}) = (e_{11}e_{10})e_{01} = [e_{10}(e_{01}e_{10})]e_{01} = e_{11}$$

and so $e_{01}e_{10} = e_{00} \neq 0$. But

$$e_{00}^2 = (e_{00}e_{01})e_{10} = [(e_{01}e_{10})e_{01}]e_{10} = (e_{01}e_{11})e_{10} = e_{01}e_{10} = e_{00}$$

is an idempotent of C_{00} and must be its unity quantity.

We now use Lemma 3 with $\mathsf{B}_i = \mathsf{C}_{ii} \neq 0$ and see that $\mathsf{C}_{11}\mathsf{C}_{10} = \mathsf{C}_{10}\mathsf{C}_{00} = \mathsf{C}_{10}$, $\mathsf{C}_{01}\mathsf{C}_{11} = \mathsf{C}_{00}\mathsf{C}_{01} = \mathsf{C}_{01}$. The fact that $\mathsf{C}_{01} = \mathsf{C}_{00}\mathsf{C}_{01}$ implies that $e_{00}x_{01} = x_{01}$

for every x_{01} of C_{01}. Similarly $x_{10}e_{00} = x_{10}$ for every x_{10} of C_{10}. It is now trivial to see that $e = e_{11} + e_{00}$ is the unity quantity of C.

The mapping

$$x_{11} \to x_{11}T = e_{01}(x_{11}e_{10}) = (e_{01}x_{11})e_{10}$$

is an isomorphism of C_{11} *onto* C_{00} such that

$$y_{10}(x_{11}T) = x_{11}y_{10}, \quad (x_{11}T)z_{01} = z_{01}x_{11}$$

for every x_{11} of C_{11}, y_{10} of C_{10} and z_{01} of C_{01}. Indeed we compute

$$\begin{aligned} y_{10}[e_{01}(x_{11}e_{10})] &= (y_{10}e_{01})(x_{11}e_{10}) + [y_{10}(x_{11}e_{10})]e_{01} = x_{11}[(y_{10}e_{01})e_{10} + (y_{10}e_{10})e_{01}] \\ &= x_{11}[y_{10}(e_{01}e_{10} + e_{10}e_{01})] = x_{11}y_{10}. \end{aligned}$$

Similarly $w_{01}x_{11} = (x_{11}T)w_{01}$. Also

$$\begin{aligned} (x_{11}T)(y_{11}T) &= [e_{01}(x_{11}e_{10})](y_{11}T) = e_{01}[(x_{11}e_{10})(y_{11}T)] \\ &= e_{01}[y_{11}(x_{11}e_{10})] = e_{01}[(x_{11}y_{11})e_{10}] = (x_{11}y_{11})T. \end{aligned}$$

Since C_{11} and C_{00} are fields, this proves that T is an isomorphism of C_{11} onto C_{00}. Actually T has an inverse given by $x_{11} = e_{10}(x_{00}e_{01}) = e_{10}y_{01}$ since then

$$x_{11}T = [e_{01}(e_{10}y_{01})]e_{10} = [(e_{01}e_{10})y_{01} + (e_{01}y_{01})e_{10}]e_{10} = (x_{00}e_{01})e_{10} = x_{00},$$

a result following from $(z_{10}e_{10})e_{10} = z_{10}e_{10}^2 = 0$ and

$$(x_{00}e_{01})e_{10} + (x_{00}e_{10})e_{01} = (x_{00}e_{01})e_{10} = x_{00}e_{00} = x_{00}.$$

We now show that the set Z of all elements $z = z_{11} + z_{11}T$ is contained[4] in the centre of C. Indeed $zy_{ii} = y_{ii}z$ for every y_{ii} of C_{ii} trivially. Also

$$zy_{10} = z_{11}y_{10} = y_{10}(z_{11}T) = y_{10}z, \quad zy = yz$$

for every y of C. Since $\mathsf{Z} = \mathsf{C}_{11} + \mathsf{C}_{00}$ we know that the associators (z, x, y) with x and y in components C_{ij} are zero unless possibly when $x = x_{ij}$ and $y = y_{ij}$ are in the same C_{ij} $(i \neq j)$. But

$$\begin{gathered} [(z_{11} + z_{11}T)x_{10}]y_{10} = (z_{11}x_{10})y_{10}, \\ (z_{11} + z_{11}T)(x_{10}y_{10}) = (z_{11}T)(x_{10}y_{10}) = [x_{10}(z_{11}T)]y_{10} = (z_{11}x_{10})y_{10} \end{gathered}$$

as desired.

By our construction, $\mathsf{C}_{11} = e_{11}\mathsf{Z}$ and $\mathsf{C}_{00} = e_{00}\mathsf{Z}$ are one-dimensional algebras over Z. We also note that since $e_{10}e_{01} = e_{10}(f_{10}g_{10}) = e_{11}$ we may use (15) to obtain $g_{10}(e_{10}f_{10}) = f_{10}(g_{10}e_{10}) = e_{11}$. Put

$$e_{10}f_{10} = g_{01}, \quad g_{10}e_{10} = f_{01}$$

and obtain (1). Then

$$g_{01}g_{10} = (e_{10}f_{10})g_{10} = (f_{10}g_{10})e_{10} = e_{00} = (g_{10}e_{10})f_{10} = f_{01}f_{10}$$

and we have (3). Now $e_{10}g_{01} = e_{10}(e_{10}f_{10}) = 0$, $e_{10}f_{01} = e_{10}(g_{10}e_{10}) = -e_{10}(e_{10}g_{10}) = 0$ since $e_{10}^2 = 0$. Similarly

[4]If C has characteristic not two or three the property $zy = yz$ implies that $(z, x, y) = 0$. However our proof is so arranged that $(z, x, y) = 0$ is quite trivial.

$$f_{10}g_{01} = f_{10}e_{01} = g_{10}e_{01} = g_{10}f_{01} = 0,$$
$$g_{01}e_{10} = f_{01}e_{10} = g_{01}f_{10} = e_{01}f_{10} = e_{01}g_{10} = f_{01}g_{10} = 0$$

and we have completed a proof which shows that (4) holds. The computation

$$e_{01}(g_{10}e_{10}) + g_{10}(e_{01}e_{10}) = e_{01}f_{01} + g_{10} = (e_{01}g_{10} + g_{10}e_{01})e_{10} = 0$$

yields $g_{10} = f_{01}e_{01}$. The remaining formulae of (2) are derived similarly.

We have now shown that **C** contains an algebra **D** over **Z** with the multiplication table given by (1)-(4). It remains only to show that e_{ij}, f_{ij}, g_{ij} are linearly independent over **F** and that these elements form a basis of $\mathbf{C}_{ij}$ over **Z** in order to prove that **D** is the eight-dimensional Cayley algebra over **Z** and that **C** = **D**.

LEMMA 7. *Let $h_{ij}h_{ji} = e_{ii}$ so that $h_{ji}h_{ij} = e_{jj}$. Then $x_{ij}h_{ij} = 0$ if and only if $x_{ij} = \alpha h_{ij}$ for α in* **Z**.

We have $x_{ij}(e_{ii} + e_{jj}) = x_{ij} = (x_{ij}h_{ij})h_{ji} + (x_{ij}h_{ji})h_{ij}$. If $x_{ij}h_{ij} = 0$ then $x_{ij} = \alpha h_{ij}$ with $x_{ij}h_{ji} = \alpha e_{ii}$ and α in **Z**. The converse follows from $h_{ij}^2 = 0$.

LEMMA 8. *Let $x_{ij}e_{ji} = x_{ij}f_{ji} = x_{ij}g_{ji} = 0$. Then $x_{ij} = 0$.*

If $x_{ij}h_{ji} = \alpha e_{ii}$ and $h_{ji}x_{ij} = \beta e_{jj}$ then

$$h_{ji}(x_{ij}h_{ji}) = \alpha h_{ji} = (h_{ji}x_{ij})h_{ji} = \beta h_{ji}.$$

If $h_{ji} \neq 0$ then $\alpha = \beta$. Now $x_{ij}e_{ij} = \pm x_{ij}(f_{ji}g_{ji})$ by (1) and (2) and so

$$x_{ij}e_{ij} = \pm [g_{ji}(x_{ij}f_{ji}) - (g_{ji}x_{ij})f_{ji}]$$

by (14). It follows that $x_{ij}e_{ij} = 0$ and that $x_{ij} = \alpha e_{ij}$. Similarly $x_{ij} = \beta f_{ij}$. But if $\alpha \neq 0$ we have

$$\alpha e_{ij}f_{ij} = \pm \alpha g_{ij} = \beta f_{ij}^2 = 0$$

contrary to hypothesis. Hence $\alpha = 0$, $x_{ij} = 0$.

It is evident that the proof above implies that $f_{ij} \neq \alpha e_{ij}$ for α in **Z**. If $g_{ij} = \alpha e_{ij} + \beta f_{ij}$ then

$$g_{ij}e_{ij} = \pm f_{ij} = \beta f_{ij}e_{ij} = \pm \beta g_{ij}$$

which has been shown to be impossible. We have shown that **D** is an eight-dimensional algebra.

We now let $x_{ij}e_{ji} = \alpha e_{ii}$, $x_{ij}f_{ji} = \beta e_{ii}$, $x_{ij}g_{ji} = \gamma e_{ii}$ for α, β, γ in **Z**. Then $y_{ij} = x_{ij} - (\alpha e_{ij} + \beta f_{ij} + \gamma g_{ij})$ has the property that

$$y_{ij}e_{ji} = (\alpha - \alpha)e_{ii} = 0,$$
$$y_{ij}f_{ji} = (\beta - \beta)e_{ii} = 0,$$
$$y_{ij}g_{ji} = (\gamma - \gamma)e_{ii} = 0$$

and so $y_{ij} = 0$ by Lemma 8. This completes our proof.

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A SYMMETRIC PROOF OF THE RIEMANN-ROCH THEOREM, AND A NEW FORM OF THE UNIT THEOREM

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Introduction. Let $F(z, u)$ denote

$$(1) \qquad u^n + u^{n-1}F_1(z) + \ldots + F_n(z),$$

where $F_1(z), \ldots, F_n(z)$ are rational functions of z with complex coefficients. We shall speak of $F(z, u) = 0$ as the fundamental algebraic equation and shall adopt z as the independent variable and u as the dependent, except in § 4, where we use x and y instead of them, and where it is understood that x and y are connected birationally with z and u. We shall say that the fundamental equation is reducible in the field obtained by adjoining z to the totality of complex numbers, provided $F(z, u)$ is the product of two factors $H(z, u)$ and $K(z, u)$ of the same form as itself and beginning with the terms u^h and u^k of positive degrees. If no such factorization of $F(z, u)$ exists, we shall say that the fundamental equation is irreducible in the stated field. Nothing is lost by excluding the case where $F(z, u)$ has repeated factors.

The form of the Riemann-Roch Theorem [**8**, Chap. 19, § 5, I] in which z is adopted as the independent variable is

$$(2) \qquad N(\tau) + \tfrac{1}{2}\sum\sum \tau\nu = N(\bar{\tau}) + \tfrac{1}{2}\sum\sum \bar{\tau}\nu.$$

In § 4 we shall show that the theorem has invariant character, in the sense that it continues to hold when any rational function of (z, u), say x, is used as a substitute for z in playing the role of independent variable, provided x is non-constant for each of the irreducible equations contained within the fundamental algebraic equation.

In the meantime, it is necessary to state what is meant by the various items appearing in (2). We shall speak of an order basis (τ) or a divisor A. A given individual order τ refers to a given u-branch at a given z-value. The z-value is the primary thing, since it alone is responsible for the division of the expansions of a rational function of (z, u) into cycles. The only point in mentioning u derives from the fact that its expansions are certainly all different, and so it is easy to recognize from them what the cycle distribution is. If the u-branch belongs to a u-cycle made up of ν branches altogether, then the order τ must refer to each of these u-branches and must be a multiple of $1/\nu$. The totality of all individual orders τ taken for all u-branches at all z-values is the order basis (τ), where it must be understood that only a finite number of the individual orders τ are different from zero. Our concern is with rational functions of (z, u) taking orders

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for all u-branches at all z-values equal to or greater than the corresponding orders in (τ). Such rational functions of (z, u) are said to be built on the order basis (τ). The number of linearly independent rational functions of (z, u) built on the order basis (τ) is denoted by $N(\tau)$, or dimension (A), which is positive except in the case where 0 is the only rational function of (z, u) built on the order basis (τ). The sum $\sum\sum \tau\nu$ of all the individual orders τ making up the order basis (τ) is, of course, an integer and is denoted by $-n(A)$. The primary element of the summation is τ, which refers to a u-branch at a z-value, while the secondary element is $\tau\nu$, which refers to a u-cycle at a z-value. These secondary elements $\tau\nu$ are then summed for all u-cycles at all z-values. Thus far, we have indicated what is meant by the (τ) side of (2), namely

$$N(\tau) + \tfrac{1}{2}\sum\sum\tau\nu, \tag{3}$$

which we shall denote by $RR(\tau)$ and refer to as the Riemann-Roch expression for the order basis (τ). As far as the $(\bar{\tau})$ side of (2) is concerned, it is only necessary to state how $(\bar{\tau})$ is derived from (τ). Order bases (τ) and $(\bar{\tau})$ are said to be complementary, to the level of a rational function $S(z, u)$ not identically zero for any u-branch at any z-value, provided

$$\tau + \bar{\tau} = \sigma - 1 + 1/\nu \tag{4}$$

for all u-branches at all finite z-values, while

$$\tau + \bar{\tau} = \sigma + 1 + 1/\nu \tag{5}$$

for all u-branches at the infinite z-value, where (σ) is the order basis composed of the exact orders of $S(z, u)$ for all u-branches at all z-values. The value of the $(\bar{\tau})$ side of (2) is the same for all such rational functions $S(z, u)$, since both $N(\bar{\tau})$ and $\sum\sum \bar{\tau}\nu$ are the same. A first important special choice of $S(z, u)$ is the function $\partial F(z, u)/\partial u$. Since $F(z, u)$, has no repeated factors, the function $\partial F(z, u)/\partial u$ supplies an order basis, to be denoted by (μ). This is the choice of $S(z, u)$ that will be made in §§ 1, 2, 3. It owes its importance to its presence in the Lagrange Interpolation Formula for reducing a rational function $R(z, u)$, as given by

$$\sum_i \left(R(z,u)/\frac{\partial F(z,u)}{\partial u}\right)_{u=u_i} \frac{F(z, u)}{u-u_i}. \tag{6}$$

A second important special choice of $S(z, u)$ is a function which is a constant different from zero for all of the irreducible equations making up $F(z, u) = 0$. It supplies by its orders the order basis (0). This is the choice of $S(z, u)$ to be made use of in § 4. It identifies the order basis $(\bar{\tau})$ with the divisor W/A, where W is the divisor equivalent to the order basis made up of orders $-1 + 1/\nu$ for all u-branches at all finite z-values, and orders $1 + 1/\nu$ for all u-branches at the infinite z-value. A natural source of each of these individual orders is well known. Indeed, for an individual u-branch at an individual z-value the individual order presents itself as the least order we can permit a rational function of (z, u) to have there, if, on multiplying the function by dz and integrating the

resulting differential, we insist on the integral being finite there. We shall write (2) from here on in the form

$$RR(\tau) = RR(\bar{\tau}). \tag{7}$$

In [7], formula (7) was established directly for assigned complementary order bases (τ) and $(\bar{\tau})$. In [2] and [4], the issue turned on the result of depressing (τ) to (t) and correspondingly raising $(\bar{\tau})$ to $(\bar{t})$ until a stage was reached when $N(t)$ became known and $N(\bar{t})$ became zero. The present paper treats both complementary order bases (τ) and $(\bar{\tau})$ in the same general way. Indeed, we shall show in § 1 how to pass from complementary order bases (τ) and $(\bar{\tau})$ to complementary order bases (t) and $(\bar{t})$ in such a way that we can count on the equations

$$N(t) = N(\bar{t}) = 0 \text{ and } RR(t) - RR(\bar{t}) = RR(\tau) - RR(\bar{\tau})$$

holding. The resulting combination of complementary order bases (t) and $(\bar{t})$ is called the 0-0 case. In § 2, we shall obtain a lower bound for the value of $N(\tau)$ and make use of it in § 3, in combination with the results of § 1, to complete the proof of the Riemann-Roch Theorem and set up a new form of the Unit Theorem equivalent to it.

1. The 0-0 case. We have already indicated in the introduction what we aim to do in this section. Given complementary order bases (τ) and $(\bar{\tau})$, it is to find complementary order bases (t) and $(\bar{t})$ such that

$$N(t) = N(\bar{t}) = 0 \text{ and } RR(t) - RR(\bar{t}) = RR(\tau) - RR(\bar{\tau}).$$

There is nothing to do if $N(\tau) = N(\bar{\tau}) = 0$. We may suppose, therefore, that $N(\tau)$ and $N(\bar{\tau})$ are not both 0, and we shall show that a finite number of applications of a certain typical process gives us the complementary order bases (t) and $(\bar{t})$ that we are after. It will be enough to describe the first application of the process. Taking $N(\tau)$ to be positive, let us select a u-branch at some z-value so that the corresponding order τ of the order basis (τ) is taken by some of the rational functions of (z, u) built on the order basis (τ). Suppose the u-branch selected belongs to a u-cycle of ν branches. Let us replace each of these orders τ as they appear in the order basis (τ) by $\tau + 1/\nu$ to get a new order basis differing from the original order basis (τ) only in respect of the u-branch selected and the remaining u-branches of the same u-cycle, and then simply by being $1/\nu$ greater. The Riemann-Roch expression for the new order basis is, therefore, less than $RR(\tau)$ by $\frac{1}{2}$, seeing that its first term is less than $N(\tau)$ by 1, while its second term is greater than $\frac{1}{2}\sum\sum \tau\nu$ by $\frac{1}{2}$. We can be certain that the Riemann-Roch expression for the order basis complementary to the new order basis is less than $RR(\bar{\tau})$ by $\frac{1}{2}$, if we can satisfy ourselves that its first term is exactly $N(\bar{\tau})$, while its second term is $\frac{1}{2}$ less than $\frac{1}{2}\sum\sum \bar{\tau}\nu$. It is clear that the latter of these statements is true. We shall now show that the former one is also. There is no rational function of (z, u) built on the order basis complementary to the new order basis but not built on the order basis $(\bar{\tau})$. For if so, let it be multi-

plied by a rational function of (z, u) built on the order basis (τ) but taking the exact order τ for the u-branch selected at the z-value, and let the product be divided by $\partial F(z, u)/\partial u$. It is clear that this would give us a rational function of (z, u) whose order for the u-branch selected would be -1, or 1, according as the z-value was finite, or infinite. Indeed, this would be the case for all the other u-branches in the same u-cycle as well. In short, we should end up by having a rational function of (z, u) with only a single residue, which we know to be impossible. To sum up, we can say that complementary order bases (τ) and $(\bar{\tau})$ were replaced by a new order basis and its complementary order basis, and that $N(\tau)$, $N(\bar{\tau})$, and $RR(\tau) - RR(\bar{\tau})$ were replaced by $N(\tau) - 1$, $N(\bar{\tau})$, and $RR(\tau) - RR(\bar{\tau})$. Hence, after a finite number of applications of the typical process described above we arrive at complementary order bases (ι) and $(\bar{\iota})$ with the properties:

$$N(\iota) = N(\bar{\iota}) = 0 \quad \text{and} \quad RR(\iota) - RR(\bar{\iota}) = RR(\tau) - RR(\bar{\tau}),$$

as required in the 0-0 case.

2. A lower bound for $N(\tau)$. The result of the present section contrasts with that of § 1 as positive with negative, in the sense that it derives from adopting a degree of generality which the functions involved succeed in reaching, whereas in § 1 a degree of generality presented itself which was never attained by any of the functions involved. In particular, we aim to show that

$$N(\tau) > n + \sum\sum \alpha. \tag{8}$$

The fundamental exponents (α) derive from the order basis (τ) and from the use of u as dependent variable. Indeed, the derivation takes place locally, that is for each z-value taken by itself. In other words, (α) derives from (τ), where the single brackets in each case refer to the individual z-value we wish to consider. In particular, let (τ) be the part of (τ) that refers to the individual z-value $z = a$. Moreover, let

$$w_{n-1}(z,u), \ldots, w_0(z,u) \tag{9}$$

be a local function basis for all the rational functions of (z, u) built on (τ), or, in other words, let all these rational functions of (z, u) be just those of the form

$$P_{n-1}(z)w_{n-1}(z,u) + \ldots + P_0(z)w_0(z,u),$$

where $P_{n-1}(z), \ldots, P_0(z)$ are rational functions of z regular at $z = a$. Indeed, the local function basis in (9) may be normalized so that the functions in order may be of degrees $n - 1, \ldots, 0$ in u, and, furthermore, so that the leading coefficient is just a power of $z - a$ in each case. This makes the functions in (9) start off with the terms

$$\frac{u^{n-1}}{(z-a)^{\alpha_{n-1}}}, \ldots, \frac{1}{(z-a)^{\alpha_0}}, \tag{10}$$

which puts in evidence the local fundamental exponents

$$\alpha_{n-1}, \ldots, \alpha_0, \tag{11}$$

which we denote collectively by (α). When the infinite z-value is chosen instead of $z = a$ as the individual z-value, the corresponding discussion applies, with, however, $1/z$ replacing $z - a$ as element.

It is natural to exhibit a local function basis (9) as a matrix. Each row of the matrix has as elements the coefficients of $u^{n-1}, \ldots, 1$ for any one of the functions in (9), while each column has as elements the coefficients of the functions in (9) for any one of $u^{n-1}, \ldots, 1$. By considering the matrices of equivalent local function bases corresponding to a local order basis (τ), it is proven in [4] that

$$\sum \tau\nu + \sum \alpha = \tfrac{1}{2} \sum (\mu - 1 + 1/\nu)\nu, \tag{12}$$

which, on being quoted for all z-values simultaneously and the results totalled up, enables us to write (8) in the form

$$N(\tau) > n - \sum\sum \tau\nu - \tfrac{1}{2} \sum\sum (\nu - 1). \tag{13}$$

For convenience of proof, however, (8) is to be preferred to (13).

We shall first show that the proof of (8) may be reduced to the case where u is without poles for all finite z-values. Given an order basis (τ), let (α) denote the fundamental exponents resulting from the use of u as dependent variable. When $u(z - a)$ is used as dependent variable instead of u, the fundamental exponents remain the same as before, except for the finite z-value $z = a$ and the infinite z-value. For $z = a$ they have to be increased by $n - 1, \ldots, 0$ over what they were originally, whereas for the infinite z-value they have to be decreased by these same amounts. In other words, the total $\sum\sum \alpha$ is not changed. Hence, the adoption of $u(z - a)$ as dependent variable instead of u produces no change in either side of (8). We can say, therefore, that if (8) is valid in either case, so is it in the other as well. It suffices, therefore, to prove (8) on the assumption that u is without poles for all finite z-values, since the preceding argument can be applied until a dependent variable is obtained whose only poles are at the infinite z-value.

Most of the simplification involved in dealing with the case where u is without poles for all finite z-values takes place at the local level and is due to special properties of the normalized function basis in (9) corresponding to the local order basis (τ) for the z-value $z = a$. These properties, which will be found fully discussed in [4], are first that $\alpha_{n-1}, \ldots, \alpha_0$ are monotone decreasing, and second that when $w_i(z, u)$ in (9) is normalized and written in the form

$$\frac{u^i + u^{i-1}H^i_{i-1}(z) + \ldots + H^i_0(z)}{(z - a)^{\alpha_i}}, \tag{14}$$

each $H^i_j(z)$ is regular at $z = a$ and may be taken to be a polynomial of degree less than $\alpha_i - \alpha_j$, in which case its form will be unique. In the sequel, we shall always make use of the unique polynomial form for $H^i_j(z)$.

Local function bases corresponding to local order bases at all finite z-values as they are involved in (τ) can be combined to form a function basis

$$W_{n-1}(z,u), \ldots, W_0(z,u) \tag{15}$$

for all rational functions of (z, u) built simultaneously on all the constituent local order bases (τ) of the order basis $\boldsymbol{(\tau)}$ at all finite z-values. All but a finite number of these local function bases will be $u^{n-1}, \ldots, 1$. The exceptions will be associated with individual finite z-values $z = a$, $z = b, \ldots$ and will have the form of the local function basis in (9), or will be patterned after it, with the element $z - a$ being replaced by $z - b, \ldots$. Naturally, this combined function basis (15) serves locally just as well as the local function basis $u^{n-1}, \ldots, 1$ wherever this applies, or as the local function basis in (9), or others patterned after it, wherever these apply. This is due to the fact that rational functions of (z, u) exist taking simultaneously the precise orders of $\boldsymbol{(\tau)}$ at all finite z-values, which is made possible through the circumstance that we have left complete freedom as to orders at the infinite z-value.

The coefficients of $u^{n-1}, \ldots, 1$ in $W_{n-1}(z, u)$ are all of fixed orders at the infinite z-value. In particular, the order of the coefficient of u^{n-1} at the infinite z-value is $\sum' a_{n-1}$ exactly, where the prime implies that the summation ranges over all the finite z-values but does not extend to the infinite z-value. The same type of remark applies to each of the remaining functions in (15). The final one is that $W_0(z, u)$ has the exact order $\sum' a_0$ at the infinite z-value, where the prime applies as already stated.

The rational function of (z, u),

$$P_{n-1}(z)W_{n-1}(z,u) + \ldots + P_0(z)W_0(z,u), \tag{16}$$

in which $P_{n-1}(z), \ldots, P_0(z)$ are arbitrary polynomials of suitably limited degrees, will serve as a sufficiently general rational function of (z, u) built simultaneously on all the constituent local order bases (τ) of the order basis $\boldsymbol{(\tau)}$ at all finite z-values. This general function (16) can, of course, be converted into the general rational function of (z, u) built on the order basis $\boldsymbol{(\tau)}$ by applying to it the conditions necessary to insure that it is also built on the constituent local order basis (τ) of the order basis $\boldsymbol{(\tau)}$ at the infinite z-value. We are safe in limiting in suitable fashion the degrees of the polynomials $P_{n-1}(z), \ldots, P_0(z)$, seeing that even the general rational function of (z, u) built on the constituent local order basis (τ) of the order basis $\boldsymbol{(\tau)}$ at the infinite z-value does not contain arbitrarily large powers of z. In other words, since the application of the conditions insuring that the function is built on the constituent local order basis (τ) of the order basis $\boldsymbol{(\tau)}$ at the infinite z-value will require the coefficients of all powers of z beyond a certain degree to vanish, we run no risk in taking them to be zero at the start. With this in mind, we shall choose

$$P_{n-1}(z), \ldots, P_0(z) \tag{17}$$

as arbitrary polynomials of degrees

$$\delta_{n-1} + \sum' a_{n-1}, \ldots, \delta_0 + \sum' a_0,$$

where the dash implies that each summation ranges over all finite z-values but does not extend to the infinite z-value. The number of arbitrary constants

appearing in (17), or, what is the same thing, in the parent function (16), is seen to be

$$\sum \delta + n + \sum\sum{}' a, \tag{18}$$

where the prime implies that the double summation ranges over all finite z-values but does not extend to the infinite z-value.

Up to the present, we have merely said that the numbers (δ) need not be chosen arbitrarily large. It is necessary, however, to indicate how we propose to limit them. This we shall attend to in two separate stages.

In the first place, we observe that

$$P_{n-1}(z)W_{n-1}(z,u), \ldots, P_0(z)W_0(z,u)$$

are of degrees $\delta_{n-1}, \ldots, \delta_0$ in z, and, indeed, that these degrees attach to the coefficients of

$$u^{n-1} \text{ in } P_{n-1}(z)W_{n-1}(z,u), \ldots, \ u^0 \text{ in } P_0(z)W_0(z,u)$$

respectively. We wish to be able to write (16) in the form

$$z^{\delta_{n-1}}Q_{n-1}(1/z)u^{n-1} + \ldots + z^{\delta_0}Q_0(1/z), \tag{19}$$

in which $Q_{n-1}(1/z), \ldots, Q_0(1/z)$ are all rational functions of z regular at the infinite z-value and all containing arbitrary constants as their initial terms when they are expanded in powers of the element $1/z$. It is clear that this is achieved by insuring that the degree in z in the coefficient of u^i in $P_i(z)W_i(z, u)$, namely, δ_i, exceeds the degree in z of the coefficient of u^i in

$$P_{n-1}(z)W_{n-1}(z,u) + \ldots + P_{i+1}(z)W_{i+1}(z,u),$$

and this for all the cases $i = n - 2, \ldots, 0$. These inequalities range, therefore, from the first of them, namely, that δ_{n-2} should exceed the degree in z of the coefficient of u^{n-2} in $P_{n-1}(z)W_{n-1}(z, u)$ to the last of them, namely, that δ_0 should exceed the degree in z of the coefficient of u^0 in

$$P_{n-1}(z)W_{n-1}(z,u) + \ldots + P_1(z)W_1(z,u).$$

Let now the local function basis equivalent to the constituent local order basis (τ) of the order basis (τ) at the infinite z-value be denoted by

$$w^{\infty}_{n-1}(z,u), \ldots, w^{\infty}_0(z,u). \tag{20}$$

This local function basis follows the pattern of the local function basis in (9), with, however, $1/z$ replacing $z - a$ as element. Let the fundamental exponents associated with this local function basis and with the use of u as dependent variable be denoted by

$$\alpha^{\infty}_{n-1}, \ldots, \alpha^{\infty}_0.$$

The form of the general rational function of (z, u) built on the constituent local order basis (τ) of the order basis (τ) at the infinite z-value is, therefore,

$$P^{\infty}_{n-1}(1/z)w^{\infty}_{n-1}(z,u) + \ldots + P^{\infty}_0(1/z)w^{\infty}_0(z,u), \tag{21}$$

in which

$$P^{\infty}_{n-1}(1/z), \ldots, P^{\infty}_{0}(1/z)$$

are all rational functions of z regular at the infinite z-value. We wish to be able to write (19), and hence (16) also, in the form (21) and must prepare for it by taking account of a second group of inequalities involving the numbers (δ). The identification of (19) with (21) proceeds naturally and simply if we take, in the first place,

$$\delta_i > \alpha^{\infty}_i \qquad (i = n - 1, \ldots, 0)$$

and, in the second place, δ_i greater than the maximum degree in z of the coefficient of u^i in any of the functions

$$z^{\delta_{n-1} - \alpha^{\infty}_{n-1}} w^{\infty}_{n-1}(z,u), \ldots, z^{\delta_{i+1} - \alpha^{\infty}_{i+1}} w^{\infty}_{i+1}(z,u)$$

for $i = n - 2, \ldots, 0$.

We are now in a position to take the final step in the proof of inequality (8). The number of linearly independent conditions involved in identifying (19) with (21) is at most

$$\sum \delta - \sum \alpha^{\infty}.$$

The proof is based on the fact that (19), even without conditions, can be written in the form of (21), except that where the rational functions

$$P^{\infty}_{n-1}(1/z), \ldots, P^{\infty}_{0}(1/z)$$

of (21) are without poles at the infinite z-value the corresponding rational functions of (19) can have poles of orders

$$\delta_{n-1} - \alpha^{\infty}_{n-1}, \ldots, \delta_0 - \alpha^{\infty}_0$$

at most. It is these poles that have to be made to disappear in the process of identifying (19) with (21). Hence, the number of linearly independent conditions required to effect this disappearance is not more than

$$\sum \delta - \sum \alpha^{\infty},$$

and when it is subtracted from the number in (18) of arbitrary constants involved in (19) at the outset we have left over not less than

$$n + \sum\sum \alpha$$

arbitrary constants. However, since we have exactly $N(\tau)$ arbitrary constants left over, we must conclude that

$$N(\tau) > n + \sum\sum \alpha,$$

which is inequality (8).

3. The Riemann-Roch Theorem and a new form of the Unit Theorem. Where (α) is the set of fundamental exponents associated with a local order basis (τ) at a given z-value and based on the use of u as dependent variable and where $(\bar{\alpha})$ is the set of fundamental exponents associated with the complemen-

tary local order basis $(\bar{\tau})$ at the given z-value and based on the use of u as dependent variable, it follows from (12) and the corresponding result for $(\bar{\tau})$ that

$$\sum \alpha + \sum \bar{\alpha} = 0, \tag{22}$$

where the equations relating complementary orders to one another are of the form

$$\tau + \bar{\tau} = \mu - 1 + 1/\nu.$$

If, however, the local order basis $(\bar{\tau})$ is exactly 2 more than enough in each of its orders to be complementary in the above sense to the local order basis (τ), the right side of (22) becomes $-2n$. But, this is precisely what does happen when (τ) and $(\bar{\tau})$ are local order bases taken from complementary order bases (τ) and $(\bar{\tau})$ at the infinite z-value, always supposing that $\partial F(z, u)/\partial u$ is the level function made use of to relate the orders of (τ) and $(\bar{\tau})$. Where (α) and $(\bar{\alpha})$ are the fundamental exponents associated with complementary order bases (τ) and $(\bar{\tau})$, we have, therefore,

$$\sum\sum \alpha + \sum\sum \bar{\alpha} = -2n. \tag{23}$$

Let us now apply inequality (8) to the 0-0 case, or, what is the same thing, let us make a joint application of §§ 1 and 2. Beginning with complementary order bases (τ) and $(\bar{\tau})$, let us make use of § 1 to obtain complementary order bases (t) and $(\bar{t})$ which it provides, and let us denote the fundamental exponents of (t) and $(\bar{t})$ by (α) and $(\bar{\alpha})$. Now applying inequality (8) of § 2 to (t) and $(\bar{t})$ separately, we find that

$$\begin{cases} 0 \geqslant n + \sum\sum \alpha, \\ 0 \geqslant n + \sum\sum \bar{\alpha}. \end{cases} \tag{24}$$

When the inequalities in (24) are added and the result compared with (23), we see that the equality sign applies in both cases in (24). In other words,

$$\sum\sum \alpha = -n = \sum\sum \bar{\alpha}. \tag{25}$$

But, we also have

$$\sum\sum \alpha + \sum\sum t\nu = \tfrac{1}{2} \sum\sum (\mu - 1 + 1/\nu)\nu = \sum\sum \bar{\alpha} + \sum\sum \bar{t}\nu, \tag{26}$$

as appears from quoting (12) for all the individual constituent local order bases of (t) and $(\bar{t})$ separately and adding up the results in each case. It follows, therefore, from (25) and (26), that

$$\sum\sum t\nu = \sum\sum \bar{t}\nu$$

and from this, in turn, that

$$RR(t) = RR(\bar{t}).$$

Hence, by § 1, we have that

$$RR(\tau) = RR(\bar{\tau}),$$

which is the statement of the Riemann-Roch Theorem for complementary order bases (τ) and $(\bar{\tau})$.

The new form of the Unit Theorem replaces the inequality (8) by an equation, namely by

$$N(\tau) = n + \sum\sum \alpha + N(\bar{\tau}), \tag{27}$$

where (α) denotes the fundamental exponents associated with (τ) and depending on the use of u as dependent variable. We see at once that

$$n + \sum\sum \alpha = \tfrac{1}{2}\sum\sum \bar{\tau}\nu - \tfrac{1}{2}\sum\sum \tau\nu, \tag{28}$$

since each side reduces to

$$\tfrac{1}{2}\sum\sum \alpha - \tfrac{1}{2}\sum\sum \bar{\alpha}.$$

The Unit Theorem says that the difference

$$N(\tau) - N(\bar{\tau})$$

is equal to the left side of (28), while the Riemann-Roch Theorem says that it equals the right side. The two theorems are, therefore, equivalent.

The original form of the Unit Theorem was that

$$N(\tau_-) - N(\tau) + N(\bar{\tau}) - N(\bar{\tau}_+) = 1. \tag{29}$$

Here (τ) and $(\bar{\tau})$ were complementary order bases, and (τ_-) and $(\bar{\tau}_+)$ were also. It was understood that (τ_-) was obtained from (τ) by depressing a single one of its individual orders by the minimum amount $1/\nu$, while $(\bar{\tau}_+)$ was obtained from $(\bar{\tau})$ by raising the corresponding one of its individual orders by $1/\nu$. It follows from the new form of the Unit Theorem that for a decrease of 1 in any cycle order in (τ) there is either an increase of 1 in $N(\tau)$ but no change in $N(\bar{\tau})$ or no change in $N(\tau)$ but a decrease of 1 in $N(\bar{\tau})$, since (12) shows that $\sum\sum \alpha$ increases by 1. In other words, the new form of the Unit Theorem implies the original form. But conversely, the original form implies the new form, since it certainly implies the Riemann-Roch Theorem, which is equivalent to the new form.

The new form of the Unit Theorem is significant, in that it associates quantities (α) determined by considering all z-values one at a time with quantities $N(\tau)$ and $N(\bar{\tau})$ determined by considering all z-values simultaneously. For that matter, the same remark applies to the Riemann-Roch Theorem itself, where it is (τ) instead of (α) which is determined by dealing with all z-values one at a time.

It is clear from (23) that the statement in (27) of the new form of the Unit Theorem simply repeats itself when the roles of (τ) and $(\bar{\tau})$ are interchanged.

4. The invariant character of the Riemann-Roch Theorem. If the fundamental algebraic equation $F(z, u) = 0$ is irreducible, and if x is a non-constant rational function of (z, u), there is a well-known routine for setting up a rational function of (z, u), say y, so that z and u are both expressible as rational functions of (x, y), and, moreover, the algebraic equation obtained by eliminating z and u, say $G(x, y) = 0$, is irreducible. If, however, $F(z, u) = 0$ is reducible, and if x is a non-constant for each of the ρ irreducible equations making up $F(z, u) = 0$,

it is clear that y can still be found so that the new pair (x, y) are birationally equivalent to the original pair (z, u), and, moreover, the algebraic equation obtained by eliminating z and u, say $G(x, y) = 0$, breaks up into ρ irreducible equations. To see this it is only necessary to make use of

$$(30) \qquad \sum_j \left(R(z,u) / \frac{F(z,u)}{f_j(z,u)} \right)_{f_j(z,u)=0} \frac{F(z,u)}{f_j(z,u)}$$

as the reduced form of a given rational function $R(z, u)$, where $F(z, u)$ is the product of ρ irreducible factors

$$f_1(z,u), \ldots f_\rho(z,u),$$

all different from one another. Here the first factor of the typical summand in (30) denotes the polynomial in u, with coefficients rational functions of z, obtained on reducing

$$R(z,u) / \frac{F(z,u)}{f_j(z,u)}$$

with respect to the irreducible equation $f_j(z, u) = 0$. That is, (30) is composed of ρ summands, formed as j ranges over $1, \ldots, \rho$. Each summand is identically 0 for $\rho - 1$ of the irreducible equations making up $F(z, u) = 0$ but ordinarily is not identically 0 for the particular irreducible equation involved in the reduction of its first factor.

Before we can say that the Riemann-Roch Theorem is invariant, [6, § 25] we have to see that it applies to $G(x, y) = 0$ as much as to $F(z, u) = 0$. That is there is to be no change in the Riemann-Roch expression when we shift from an order basis (τ) relative to $F(z, u) = 0$ to the corresponding order basis (t) relative to $G(x, y) = 0$. Besides, when we shift from complementary order bases (τ) and $(\bar{\tau})$ relative to $F(z, u) = 0$ to order bases (t) and $(\bar{t})$ relative to $G(x, y) = 0$, this latter pair of order bases is to be complementary relative to $G(x, y) = 0$, the maintenance if the complementary property involving nothing beyond a natural change from the function used as level for $F(z, u) = 0$ to the one used as level for $G(x, y) = 0$. It will be convenient to adopt x as the independent variable and y as the dependent variable when we are making use of $G(x, y) = 0$ as the fundamental algebraic equation.

When we speak of a cycle about a z-value, we shall continue to use ν generically to denote the number of its branches, and this whether we refer to a finite z-value $z = a$ or the infinite z-value as centre. In the same way, when we speak of a cycle about an x-value, we shall use ω generically to denote the number of its branches, and this whether we refer to a finite x-value $x = \alpha$ or the infinite x-value as centre. All types of correspondence between cycles about z-values and cycles about x-values are given, to a first approximation, by the following:

$$
(31)\qquad
\begin{aligned}
(z-a)^{\omega} &= K(x-\mathfrak{a})^{\nu},\\
(z-a)^{\omega} &= K\left(\frac{1}{x}\right)^{\nu},\\
\left(\frac{1}{z}\right)^{\omega} &= K(x-\mathfrak{a})^{\nu},\\
\left(\frac{1}{z}\right)^{\omega} &= K\left(\frac{1}{x}\right)^{\nu}.
\end{aligned}
$$

In each case K is, to a first approximation, a constant different from zero.

We shall now verify that the first requirement for the invariance of the Riemann-Roch Theorem is met. In the first case of (31), let a rational function have order $\tau = \lambda/\nu$ for each of the ν branches of the cycle in question about the finite z-value $z = a$. This same rational function will have order $t = \lambda/\omega$ for each of the ω branches of the cycle in question about the finite x-value $x = \mathfrak{a}$. In other words, from $\tau = \lambda/\nu$ for each of the ν branches of the cycle in question about the finite z-value $z = a$ we deduce $t = \lambda/\omega$ for each of the ω branches of the cycle in question about the finite x-value $x = \mathfrak{a}$, and we observe that $\tau\nu = t\omega$. The same type of discussion applies to the remaining cases of (31). An order basis (τ) relative to $F(z, u) = 0$ converts, therefore, into an order basis (t) relative to $G(x, y) = 0$, and, besides,

$$\sum\sum \tau\nu = \sum\sum t\omega.$$

Furthermore, on account of the birationality connecting the pairs (z, u) and (x, y) with one another, we see that $N(\tau)$ relative to $F(z, u) = 0$ is the same thing as $N(t)$ relative to $G(x, y) = 0$. That is, the first requirement is met, since $RR(\tau)$ relative to $F(z, u) = 0$ has the same value as $RR(t)$ relative to $G(x, y) = 0$.

We must still see that the second requirement for the invariance of the Riemann-Roch Theorem is also met. Let us suppose that order bases (τ) and $(\bar{\tau})$ are complementary relative to $F(z, u) = 0$, to the level of a function $S(z, u)$, which is a constant different from zero for each of the ρ irreducible equations making up $F(z, u) = 0$, which ρ non-zero constants may be chosen to range all the way from being all the same to being all different. We wish to show that order bases (t) and $(\bar{t})$ are complementary relative to $G(x, y) = 0$ to the level of the function $S(z, u)dx/dz$. A direct verification of this can be easily made. (*Cf.*[5, Chap. 2, § 5].) In the first case of (31) let $\tau = \lambda/\nu$ and $\bar{\tau} = -1 + (1-\lambda)/\nu$ be the individual orders taken from (τ) and $(\bar{\tau})$ for the cycle in question about the finite z-value $z = a$. Then

$$t = \frac{\lambda}{\omega} \text{ and } \bar{t} = -\frac{\nu}{\omega} + \frac{1-\lambda}{\omega},$$

while the order of dx/dz is $1 - \nu/\omega$, all relative to the cycle in question about the finite x-value $x = \mathfrak{a}$. In other words,

$$t + \bar{t} = \left(1 - \frac{\nu}{\omega}\right) - 1 + \frac{1}{\omega},$$

which means that t and $\bar{t}$ have the complementary property for the cycle in question about the finite x-value $x = a$ to the level of the function $S(z, u)dx/dz$. The same sort of verification can be given for the remaining cases of (31). That is, the second requirement for the invariance of the Riemann-Roch Theorem is also met.

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ITERATED TRANSFORMS

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1. Introduction. In his work on Laplace and Stieltjes transforms Widder [6, ch. 8] has investigated relationships of the type

$$(1) \qquad f(x) = \int_0^\infty e^{-xt} g(t)\,dt,$$

$$(2) \qquad g(x) = \int_0^\infty e^{-xt} h(t)\,dt,$$

$$(3) \qquad f(x) = \int_0^\infty \frac{h(t)}{x+t}\,dt.$$

(1) and (2) are Laplace transforms and (3), which occurs in Stieltjes' [4] researches on continued fractions, is referred to by Widder as a Stieltjes transform. Widder also considers (3) in the more general form of a Stieltjes integral

$$f(x) = \int_0^\infty \frac{dk(t)}{x+t}.$$

These formulae bear a close resemblance to special cases of Chain transforms [1], whose theory I have developed in a previous paper. My object here is to investigate and generalize the relationships above by the methods used in Chain transform theory. For example, we prove that the factors e^{-xt} in (1) and (2) can be replaced by Laplace transforms of Fourier kernels and then show that this result can be generalized still further. In order to make use of the mean square theory of convergence we shall define a Laplace transform which is somewhat more general than the one in common use.

2. The Mellin transform. The Mellin transform [5, ch. 3], which is our main instrument of analysis, is given by

$$(4) \qquad F(s) = \int_0^\infty f(u) u^{s-1}\,du,$$

$$(5) \qquad f(u) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} F(s) u^{-s}\,ds.$$

Pairs of functions related in the manner of (4) and (5) are known as Mellin transforms of each other and will always be written in the form $f(u)$ and $F(s)$, $g(u)$ and $G(s)$, etc. Their main properties [5; §§ 3.17, 7.7, 7.8] are as follows:

2.1. If $f(u)$ belongs to $L^2(0, \infty)$, i.e.,

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$$\int_0^\infty |f(u)|^2 du$$

converges, then as a tends to infinity

$$\int_{1/a}^{a} f(u)u^{s-1}du \tag{2.11}$$

converges in mean square to $F(s)$, where $F(s)$ belongs to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$,

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} F(s)u^{-s}ds \tag{2.12}$$

converges in mean square to $f(u)$.

Conversely if $F(s)$ belongs to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ then **2.12** holds, $f(u)$ belongs to $L^2(0, \infty)$ and is related to $F(s)$ by (4).

2.2 (The Parseval theorem). If $f(u)$ and $g(u)$ both belong to $L^2(0, \infty)$, or $F(s)$ and $G(s)$ both belong to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$, then

$$\int_0^\infty f(u)g(u)du = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} F(s)G(1-s)ds. \tag{6}$$

2.3 If $f(u)$ and $F(s)$ are Mellin transforms then so are

$$f(au) \quad \text{and} \quad F(s)a^{-s},$$
$$u^a f(u) \quad \text{and} \quad F(s+a),$$
$$f(u^a) \quad \text{and} \quad \frac{1}{a}F\left(\frac{s}{a}\right).$$

2.4 A pair of Mellin transforms is given by the equations $f(u) = 1$ $(0 < u < y)$, $f(u) = 0$ $(u > y)$, and $F(s) = y^s/s$.

To illustrate these results if y is real then, treated as a function of s, y^s/s belongs to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$. Hence if $M(s)$ also belongs to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ we have from (6) and **2.4**,

$$\int_0^y m(u)du = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} M(s)\frac{y^{1-s}}{1-s}ds. \tag{7}$$

We shall, in future, write

$$\int_0^y m(u)du = m_1(y) \tag{8}$$

and all pairs of functions written in this way, e.g. $n(y)$, if it exists, and $n_1(y)$, will be related as in (8). Thus (7) becomes

$$m_1(y) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} M(s)\frac{y^{1-s}}{1-s}ds, \tag{9}$$

where $M(s)$ belongs to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$.

The function $M(s)$ plays a fundamental part in the theory of Fourier kernels. For all of these kernels, $M(s)$ is bounded on the line $s = \frac{1}{2} + i\tau$, where τ is real, but in general does not belong to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$. However, since $M(s)$ is bounded, $M(s)/(1 - s)$ belongs to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$. We can therefore deduce from **2.12** and (9) that $m_1(y)/y$ belongs to $L^2(0, \infty)$ and that its Mellin transform is $M(s)/(1 - s)$. This may be true even if $m(y)$ does not exist, as the following example illustrates.

Let $M(s) = 1$, then

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} M(s)y^{-s}ds$$

oscillates finitely and does not converge. But from (9) we have $m_1(y) = 0$ if $0 < y < 1$ and $m_1(y) = 1$ if $y > 1$. Thus $m_1(y)$, defined by (9) instead of by (8), exists although $m(y)$ does not exist.

Partly for this reason and partly because $m_1(y)/y$ belongs to $L^2(0, \infty)$ it is much more convenient to formulate Fourier transform theory in terms of $m_1(u)$ rather than $m(u)$. For these reasons we shall also define our Laplace transform of §4 in terms of $m_1(u)$ rather than in terms of $m(u)$.

3. The general Fourier transform. If

$$M(s)N(1 - s) = 1, \tag{10}$$

and $M(s)$ and $N(s)$ are both bounded on the line $s = \frac{1}{2} + i\tau$, where τ is real, and $A(x)$ belongs to $L^2(0, \infty)$, then we have the general Fourier transform

$$\begin{aligned} B(y) &= \frac{d}{dy}\int_0^\infty A(x)\frac{m_1(xy)}{x}dx, \\ A(y) &= \frac{d}{dy}\int_0^\infty B(x)\frac{n_1(xy)}{x}dx. \end{aligned} \tag{11}$$

In the course of the proof, it is shown that $B(x)$ also belongs to $L^2(0, \infty)$. The theory is given in Titchmarsh [**5**, p. 226]. The functions $m_1(x)/x$ and $n_1(x)/x$ are known as general Fourier kernels and are called symmetrical if $m_1(x) = n_1(x)$ and asymmetrical otherwise.

4. The general Laplace transform of Fourier kernels. From the asymptotic expansion of $\Gamma(s)$ we know that on the line $s = \frac{1}{2} + i\tau$, where τ is real, $\Gamma(s) = 0(e^{-\frac{1}{2}\pi|\tau|})$ and so belongs to $L^2(0, \infty)$. Also the Mellin transform of e^{-xu} is $\Gamma(s)\, x^{-s}$. Hence, if $M(s)$ belongs to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ we have by (6),

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma(s)M(1 - s)x^{-s}ds = \int_0^\infty e^{-xu}m(u)du. \tag{12}$$

The right hand side is evidently the Laplace transform of $m(u)$.

Suppose that, as in Fourier transform theory, we know only that $M(s)$ is bounded. In this case $M(s)/(1 - s)$ belongs to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ and, from (9), is the Mellin transform of $m_1(u)/u$. Hence from (6) and **2.3** we have

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Gamma(s)M(1-s)x^{-s}ds = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Gamma(s+1)\frac{M(1-s)}{s}x^{-s}ds \tag{13}$$
$$= \int_0^\infty xe^{-xu}m_1(u)du.$$

We shall call the right-hand side of (13) the general Laplace transform and write

$$\mathbf{m}(x) = \int_0^\infty xe^{-xu}m_1(u)du. \tag{14}$$

All pairs of functions related in the manner defined by (14) will be written in the form $\mathbf{m}(x)$ and $m_1(x)$, $\mathbf{n}(x)$ and $n_1(u)$, etc.

The definition (14) has two important advantages over the standard Laplace transform (12). First, it exists if $M(s)$ is bounded, whether $m(x)$ exists or not, and, secondly, it lends itself readily to the application of mean square arguments. (14) bears much the same relation to (12) as the general Fourier kernel bears to the ordinary Fourier kernel. To illustrate these remarks take $M(s) = 1$, then, from §2, $m(u)$ does not exist. But $m_1(u) = 0$ when $0 < u < 1$ and $m_1(u) = 1$ when $u > 1$. From (14) it follows that $\mathbf{m}(x) = e^{-x}$ so that any general theorems proved for general Laplace transforms will also be true for e^{-x}.

In most cases when $m(u)$ exists it can be shown, on integrating by parts, that the right-hand sides of (12) and (14) are equal. Integration by parts also shows that (14) can frequently be expressed as the Stieltjes integral

$$\int_0^\infty e^{-xu}dm_1(u),$$

which by many writers is considered to be a natural generalization of the Laplace transform. But for our purposes it is much more convenient to use the form (14) than the Stieltjes integral. This is because of the advantage gained by using $M(s)$ in the study of Laplace transforms of Fourier kernels (see (9)).

From (13) and (14) we have

$$\mathbf{m}(x) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Gamma(s)M(1-s)x^{-s}ds, \tag{14a}$$

where $\Gamma(s)\ M(1-s)$ belongs to $L^2(\frac{1}{2} - i\infty,\ \frac{1}{2} + i\infty)$. Hence, from **2.12**, $\mathbf{m}(x)$ belongs to $L^2(0, \infty)$ and its Mellin transform is $\Gamma(s)\ M(1-s)$.

If $M(s)$ and $N(s)$ satisfy (10) and are both bounded on the line $s = \frac{1}{2} + i\tau$ then $\mathbf{m}(x)$ and $\mathbf{n}(x)$ are the general Laplace transforms of a pair of conjugate Fourier kernels. We then have, from (6),

$$\int_0^\infty \mathbf{m}(ux)\mathbf{n}(u)du = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Gamma(s)M(1-s)x^{-s}\Gamma(1-s)N(s)ds$$
$$= \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{\pi}{\sin\pi s}x^{-s}ds \tag{15}$$
$$= \frac{1}{1+x}.$$

That Laplace transforms of Fourier kernels satisfy an integral equation of the type (15) was conjectured by Ramanujan [**3**, ch. 11(F)] and proved recently by Goodspeed [**2**].

5. Iteration formulae. If $M(s)$ is the Mellin transform of a symmetrical Fourier kernel then (10) becomes

$$M(s)M(1-s) = 1. \tag{16}$$

THEOREM 1. *If* (i) $M(s)$ *satisfies* (16) *and is bounded on the line* $s = \frac{1}{2} + i\tau$, (ii) *the general Laplace transform* $\mathbf{m}(x)$ *is defined by* (14), (iii) $h(x)$ *belongs to* $L^2(0, \infty)$, *and*

$$g(u) = \int_0^\infty \mathbf{m}(ut)h(t)dt, \tag{17}$$

and

$$f(u) = \int_0^\infty \mathbf{m}(ut)g(t)dt, \tag{18}$$

then

$$f(u) = \int_0^\infty \frac{h(t)}{u+t}dt \tag{19}$$

and

$$\int_0^\infty f(ut)h(t)dt = \int_0^\infty g(ut)g(t)dt. \tag{20}$$

To prove (19) we note that $\mathbf{m}(ut)$, as a function of t, and $h(t)$ both belong to $L^2(0, \infty)$ and have Mellin transforms $\Gamma(s)\, M(1-s)\, u^{-s}$ and $H(s)$ respectively ((14a) and §2). Hence, from (6) and (17), we have

$$g(u) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma(s)M(1-s)u^{-s}H(1-s)ds. \tag{21}$$

Now on the line $s = \frac{1}{2} + i\tau$, $|\Gamma(s)\, M(1-s)|$ is bounded, say with upper bound K, and $H(s)$ belongs to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ (from condition (iii) and **2.11**). Hence, on integrating along the line $s = \frac{1}{2} + i\tau$, we have

$$\int_{-\infty}^{\infty} |\Gamma(s)M(1-s)H(1-s)|^2 d\tau < K^2 \int_{-\infty}^{\infty} |H(1-s)|^2 d\tau$$

and so $\Gamma(s)\, M(1-s)\, H(1-s)$ belongs to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$. Hence, from **2.12** and (21), $g(u)$ belongs to $L^2(0, \infty)$ and its Mellin transform $G(s)$ is given by

$$G(s) = \Gamma(s)M(1-s)H(1-s). \tag{22}$$

Since $g(u)$ belongs to $L^2(0, \infty)$ we may similarly deduce from (18) that

$$F(s) = \Gamma(s)M(1-s)G(1-s). \tag{23}$$

On substituting $1 - s$ for s in (22) and eliminating $G(1-s)$ from (23) we have

$$F(s) = \Gamma(s)M(1-s)\Gamma(1-s)M(s)H(s)$$
$$= \frac{\pi}{\sin \pi s} H(s)$$

from (16). Hence

$$f(u) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} H(s)u^{-s}\frac{\pi}{\sin \pi(1-s)}ds.$$

But $\pi/(\sin \pi s)$ belongs to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ and its Mellin transform [5, p. 192 (7.7.8)] is $1/(1+u)$. From (6) we may then conclude that

$$f(u) = \int_0^\infty \frac{1}{1+t} h(ut)dt$$

$$(24) \qquad = \int_0^\infty \frac{h(t)}{u+t}dt \qquad (u > 0)$$

by a slight change of variable. This completes the proof of (19).

For the proof of (20) we see from (22) and (23) that

$$(25) \qquad F(s)H(1-s) = G(s)G(1-s).$$

Condition (iii) shows that $h(u)$ belongs to $L^2(0, \infty)$ and in the course of the proof of (24) it was shown that $f(u)$ and $g(u)$ also belong to $L^2(0, \infty)$. Hence, from **2.11**, $F(s)$, $G(s)$, and $H(s)$ all belong to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$. On multiplying (25) by u^{-s} and integrating along the line $s = \frac{1}{2} + i\tau$ from $\frac{1}{2} - i\infty$ it follows from (6) that

$$\int_0^\infty f(ut)h(t)dt = \int_0^\infty g(ut)g(t)dt.$$

This completes the proof of Theorem 1. We have already noted that we can take $\mathbf{m}(x) = e^{-x}$ and then (17) and (18) reduce to known forms [6, p. 325].

Theorem 2 deals with the case of the asymmetrical kernels. If $M(s)$ and $N(s)$ are the Mellin transforms of a pair of conjugate Fourier kernels then the functional equation (10) is satisfied. This is dealt with in

THEOREM 2. *If* (i) $M(s)$ *and* $N(s)$ *satisfy* (10) *and are both bounded on the line* $s = \frac{1}{2} + i\tau$, (ii) *the general Laplace transforms* $\mathbf{m}(x)$ *and* $\mathbf{n}(x)$ *are defined by* (14), (iii) $h(x)$ *belongs to* $L^2(0, \infty)$, *and*

$$g(u) = \int_0^\infty \mathbf{m}(ut)h(t)dt,$$

and

$$f(u) = \int_0^\infty \mathbf{n}(ut)g(t)dt,$$

then

$$f(u) = \int_0^\infty \frac{h(t)}{u+t}dt.$$

To prove this we use the same arguments as for Theorem 1. Equation (22) can be established as before, but instead of (23) we now prove

$$F(s) = \Gamma(s)N(1-s)G(1-s). \tag{26}$$

On eliminating $G(s)$ between (22) and (26), by the method of Theorem 1, and using (10) we establish once again

$$F(s) = \frac{\pi}{\sin \pi s}H(s).$$

From this we deduce (24), as in Theorem 1, and so complete the proof of Theorem 2. But if $M(s)$ is not equal to $N(s)$ then we cannot, from (22) and (26), establish a relation such as (25). Consequently, under the conditions of Theorem 2 the Parseval equation (20) does not in general exist.

6. Formal analysis. In this section I am concerned mainly with the methods by which iterated relationships such as (1), (2), and (3) can be obtained. The analysis is purely formal and difficulties, such as arise in changing the order of integration, etc., are for the moment ignored.

Consider the two equations

$$f(x) = \int_0^\infty p(xt)g(t)dt \tag{27}$$

and

$$g(x) = \int_0^\infty q(xt)h(t)dt. \tag{28}$$

On multiplying (27) by x^{s-1} we have formally

$$\begin{aligned} F(s) &= \int_0^\infty \int_0^\infty p(xt)x^{s-1}g(t)dtdx \\ &= \int_0^\infty \int_0^\infty p(u)u^{s-1}g(t)t^{-s}dtdu \\ &= P(s)G(1-s). \end{aligned} \tag{29}$$

Similarly from (28) we have

$$G(s) = Q(s)H(1-s). \tag{30}$$

On substituting $1 - s$ for s in (30) we can eliminate $G(1 - s)$ from (29) and obtain

$$F(s) = P(s)Q(1-s)H(s).$$

Hence a relation exists between $f(x)$ and $h(x)$ which, in general, is independent of $g(x)$ but which depends largely upon the nature of the quantity $P(s)\,Q(1 - s)$ and its Mellin transform. This observation is illustrated by the following examples:

If

$$f(x) = \int_0^\infty e^{-xt}(xt)^a g(t)dt \tag{31}$$

and

$$g(x) = \int_0^\infty e^{-xt}(xt)^a h(t)dt, \tag{32}$$

then

$$f(x) = \Gamma(2a+1)\int_0^\infty \frac{h(t)(xt)^a}{(x+t)^{2a+1}}dt, \tag{33}$$

where $2a + 1 > 0$ and $x > 0$. By the method just outlined we have, from (31),

$$F(s) = \Gamma(s+a)G(1-s)$$

and from (32),

$$G(s) = \Gamma(s+a)H(1-s).$$

Hence

$$F(s) = \Gamma(s+a)\Gamma(a+1-s)H(s). \tag{34}$$

But the Mellin transform [5, p. 195] of $\Gamma(a+1-s)\ \Gamma(a+s)$ is

$$\frac{\Gamma(2a+1)u^a}{(1+u)^{2a+1}}.$$

Hence from (34) and (6) we may deduce that

$$f(x) = \Gamma(2a+1)\int_0^\infty \frac{h(xt)t^a}{(1+t)^{2a+1}}dt.$$

This is finally reduced to (33) by a simple change of variable.

This analysis can be made rigorous by assuming that $h(x)$ belongs to $L^2(0, \infty)$. By the methods of Theorem 1 we can prove the following result:

THEOREM 3. *If $f(x)$ and $g(x)$ are related as in* (31) *and $g(x)$ and $h(x)$ as in* (32), *where $2a + 1 > 0$, and $h(u)$ belongs to $L^2(0, \infty)$ then $g(u)$ and $f(u)$ also belong to $L^2(0, \infty)$, $f(x)$ and $h(x)$ are related by* (33) *and also*

$$\int_0^\infty f(ut)h(t)dt = \int_0^\infty g(ut)g(t)dt. \tag{35}$$

The equations (31), (32), and (33) reduce to (1), (2), and (3) in the special case $a = 0$.

It is not difficult to generalize Theorem 3 still further. For, write

$$\mu(t) = \int_0^\infty e^{-xt}(xt)^a m(x)dx$$

and

$$\nu(t) = \int_0^\infty e^{-xt}(xt)^a n(x)dx,$$

where $m(x)$ and $n(x)$ are a pair of conjugate Fourier kernels, so that (10) is satisfied. Then we may, in general, replace (31) and (32) by

$$f(x) = \int_0^\infty \mu(xt)g(t)dt$$

and

$$g(x) = \int_0^\infty \nu(xt)h(t)dt,$$

and the relationship between $f(x)$ and $h(x)$ is still given by (33). If $m(x) = n(x)$ then we also have the Parseval equation (35), but not otherwise. This can be proved, on assuming suitable conditions, by the same arguments as are used in the proof of Theorem 1. But since this is a special case of Theorem 5, we shall omit the proof.

A second example is given by the following result: if

$$f(x) = a\int_0^\infty e^{-x^a t^a} g(t)dt$$

and

$$g(x) = a\int_0^\infty e^{-x^a t^a} h(t)dt,$$

where $a > 0$, then

$$f(x) = a\Gamma\left(\frac{1}{a}\right)\int_0^\infty \frac{h(t)}{(x^a + t^a)^{1/a}}dt,$$

and in addition (35) is true. This system reduces to (1), (2), and (3) when $a = 1$.

This set of transforms can be justified by the arguments used in Theorem 1 if we assume that $h(u)$ belongs to $L^2(0, \infty)$. The Mellin transforms required for the proof are

$$e^{-u^a} \text{ and } \frac{1}{a}\Gamma\left(\frac{s}{a}\right), \quad \frac{a\Gamma(1/a)}{(1 + u^a)^{1/a}} \text{ and } \Gamma\left(\frac{s}{a}\right)\Gamma\left(\frac{1}{a} - \frac{s}{a}\right).$$

7. General iteration formulae. For the rest of this paper we shall find it convenient to use the following terminology. We write p for $p(x)$ or $p(u)$, Mel p for the Mellin transform of $p(u)$, P for $P(s)$, and $\bar{P}$ for $P(1 - s)$. Thus

$$\text{Mel } p = P \text{ and } \overline{\text{Mel } p} = \bar{P}.$$

We shall also write

$$[p, q]\{u\} = [p, q] = \int_0^\infty p(ut)q(t)dt. \tag{36}$$

The form $[p,q]\{u\}$ will be used only when it is necessary to specify the variable u.

If $p(x)$ and $q(x)$ both belong to $L^2(0, \infty)$ then from (6) and **2.3** we have

$$[p, q] = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} Pu^{-s}\bar{Q}ds,$$

where P and Q both belong to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$. If, in addition, either P or Q is bounded on the line $s = \frac{1}{2} + i\tau$ then, as in the proof of Theorem 1, we can infer that $P\bar{Q}$ belongs to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$, hence that $[p,q]$ belongs to $L^2(0, \infty)$ and, finally, that

$$\text{(37)} \qquad \text{Mel}\,[p, q] = P\bar{Q} = \text{Mel}\,p\,\overline{\text{Mel}\,q}.$$

We can now prove

THEOREM 4. *If* (i) $p(x)$, $q(x)$, *and* $h(x)$ *all belong to* $L^2(0, \infty)$, (ii) P *and* Q *are bounded on the line* $s = \frac{1}{2} + i\tau$, *and*

$$\text{(38)} \qquad g(u) = \int_0^\infty p(ut)h(t)dt$$

and

$$\text{(39)} \qquad f(u) = \int_0^\infty q(ut)g(t)dt,$$

then

$$\text{(40)} \qquad f(u) = \int_0^\infty [p, q]\{t\}h(ut)dt.$$

If, in addition, $p(u) = q(u)$, *then we also have*

$$\text{(41)} \qquad \int_0^\infty f(ut)h(t)dt = \int_0^\infty g(ut)g(t)dt.$$

In the terminology just described conditions (38) and (39) become $g = [p,h]$ and $f = [q,g]$ and we are required to prove that $f = [h,[p,q]]$. Equation (41) can also be written in the form $[f,h] = [g,g]$.

To prove (40), apply the arguments preceding (37) to (38), using conditions (i) and (ii). We then deduce first that $G = P\bar{H}$ and secondly that $g(u)$ belongs to $L^2(0, \infty)$. We may further deduce from (39) that $F = Q\bar{G}$ and that $f(u)$ also belongs to $L^2(0, \infty)$.

We now have

$$F = Q\bar{G} = Q\overline{P\bar{H}} = Q\bar{P}H.$$

From the conditions of integrable square, some of which have been assumed and some proved, we may apply (6) and (37) to this result and rewrite it in the form $\text{Mel}\,f = \text{Mel}\,[h,r]$, where $\text{Mel}\,r = P\bar{Q} = \text{Mel}\,[p,q]$. Hence $f = [h,[p,q]]$, which is equivalent to (40).

If $p = q$ then $P = Q$ and from $G = P\bar{H}$ and $F = Q\bar{G}$ we deduce that $F\bar{H} = G\bar{G}$. From (37) this may be written in the form $\text{Mel}\,[f,h] = \text{Mel}\,[g,g]$. Hence $[f,h] = [g,g]$, which is equivalent to (41).

Theorem 4 contains the following results as special cases. When

$$p(x) = q(x) = e^{-x}$$

it reduces to equations (1), (2), and (3). When

$$p(x) = q(x) = e^{-x}x^a$$

it reduces to Theorem 3. When

$$p(x) = q(x) = ae^{-x^2}$$

it reduces to the system of equations at the end of §6.

8. The Fourier kernel transform. In Theorem 1 it was shown that the factors e^{-xt} in (1) and (2) could be replaced by the Laplace transforms of Fourier kernels. In this section we shall show that Theorem 4 is capable of an analogous generalization.

For this purpose we shall introduce two operators T_1 and T_2. Let $M = M(s)$ and $N = N(s)$ be two functions which are bounded on the line $s = \frac{1}{2} + i\tau$ and which satisfy the functional equation

$$MN = 1. \tag{42}$$

Then by using (9) we can find functions $m_1(x)$ and $n_1(x)$ which form the basis of general Fourier transforms of the type (11) [5, p. 226]. We define the operators as follows:

$$T_1 p\{x\} = T_1 p = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} Px^{-s}\bar{M}ds \tag{43}$$

and

$$T_2 q\{x\} = T_2 q = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} Qx^{-s}\bar{N}ds. \tag{44}$$

The forms $T_1p\{x\}$ and $T_2q\{x\}$ will be used only when it is necessary to specify the variable x.

As an illustration, when $M = N = 1$ and $P = Q = \Gamma(s)$ then $T_1p = T_2q = e^{-x}$.

The Parseval equation (6) is often true even when the conditions of **2.2** are not fulfilled. Assuming that it is true for (43) and (44) we should then have

$$\begin{aligned} T_1 p &= \int_0^\infty p(xt)m(t)dt, \\ T_2 q &= \int_0^\infty q(xt)n(t)dt. \end{aligned} \tag{45}$$

If, for example, $p(x) = q(x) = e^{-x}$ then T_1p and T_2q reduce to the Laplace transforms of $m(x)$ and $n(x)$ respectively.

Again, since M and N are bounded on $s = \frac{1}{2} + i\tau$, M/s and N/s both belong to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$. Hence if sP belongs to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ we may write $sPx^{-s}\bar{M}/s$ for the right-hand integrand of (43) and use (6). We can then deduce in general that

$$T_1 p = \int_0^\infty - xp'(xt)m_1(t)dt, \tag{46}$$

where $m_1(t)$ is defined by (9) and the prime denotes differentiation. Similarly, if sQ belongs to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ then we have, in general,

$$T_2 q = \int_0^\infty - xq'(xt)n_1(t)dt. \tag{47}$$

These bear the same relation to (45) as the general Laplace transform (14) bears to the ordinary Laplace transform (12). When $m(x)$ and $n(x)$ exist it is usually possible to prove that (46) and (47) reduce to (45) (on integrating by parts).

The advantage of defining T_1 and T_2 by (43) and (44) instead of by (45) or by (46) and (47) lies in the great generality of (43) and (44). Thus if $M = N = 1$, $m(t)$ and $n(t)$ do not exist; but if $p(x)$ and $q(x)$ belong to $L^2(0, \infty)$ then, from **2.12** and (43), (44), we have $T_1 p = p$ and $T_2 q = q$.

Another advantage lies in the fact that important deductions can be made from (43) and (44) with the help of reasonably simple assumptions. The most useful one from our point of view is as follows:

If $p(x)$ and $q(x)$ both belong to $L^2(0, \infty)$ and M and N are bounded on the line $s = \frac{1}{2} + i\tau$ then $T_1 p$ and $T_2 q$ also belong to $L^2(0, \infty)$ and

$$\begin{aligned} \text{Mel } T_1 p &= P\bar{M}, \\ \text{Mel } T_2 q &= Q\bar{N}. \end{aligned} \tag{48}$$

For $p(x)$ belongs to $L^2(0, \infty)$ and so, from **2.11**, P belongs to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$. Hence, since M is bounded, $P\bar{M}$ also belongs to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$. The results stated then follow from **2.12**.

We now prove our final theorem.

THEOREM 5. *If* (i) $p(x)$, $q(x)$, *and* $h(x)$ *all belong to* $L^2(0, \infty)$, (ii) P *and* Q *are bounded on the line* $s = \frac{1}{2} + i\tau$, (iii) M *and* N *are bounded on the line* $s = \frac{1}{2} + i\tau$ *and satisfy the equation* $M\bar{N} = 1$, *and*

$$g(u) = \int_0^\infty T_1 p\{ut\} h(t)dt \tag{49}$$

and

$$f(u) = \int_0^\infty T_2 q\{ut\} g(t)dt, \tag{50}$$

then

$$f(u) = \int_0^\infty [p, q]\{t\} h(ut)dt. \tag{51}$$

If, in addition, $p(u) = q(u)$ *and* $M = N$, *then we also have*

$$\int_0^\infty f(ut)h(t)dt = \int_0^\infty g(ut)g(t)dt. \tag{52}$$

We first prove (51) by means of Theorem 4. Since $p(x)$ belongs to $L^2(0, \infty)$ it follows from (48) that $T_1 p$ belongs to $L^2(0, \infty)$ and that Mel $T_1 p = P\bar{M}$. Also from conditions (ii) and (iii) it is evident that $P\bar{M}$ is bounded on the line $s = \frac{1}{2} + i\tau$. Similar remarks apply to $T_2 q$ and to Mel $T_2 q = Q\bar{N}$. Hence conditions (i) and (ii) of Theorem 4 are satisfied. On applying the results of that theorem to (49) and (50) we find that

(53) $$f(u) = \int_0^\infty [T_1 p, T_2 q]\{t\} h(ut) dt.$$

The proof is then completed if we can show that $[T_1p, T_2q] = [p, q]$.

Since $p(x)$, $q(x)$, T_1p, and T_2q all belong to $L^2(0, \infty)$ it follows from (37) that

$$\begin{aligned} \text{Mel}\,[T_1 p, T_2 q] &= \text{Mel}\, T_1 p \,.\, \overline{\text{Mel}\, T_2 q} \\ &= P\bar{M} \,.\, \bar{Q}N && \text{from (48)} \\ &= P\bar{Q} && \text{from condition (iii) above} \\ &= \text{Mel}\,[p, q] \end{aligned}$$

from (37) again. Hence

$$[T_1 p, T_2 q] = [p, q]$$

and the proof of (51) is completed.

To prove (52), from (6) and (49) we have

$$G = \text{Mel}\, T_1 p \,.\, \bar{H} = P\bar{M}\bar{H}$$

and from (6) and (50) we have

$$F = \text{Mel}\, T_2 q \,.\, \bar{G} = Q\bar{N}\bar{G}.$$

But we now have two extra conditions: $p(x) = q(x)$, from which we derive $P = Q$, and $M = N$. Hence $F\bar{H} = G\bar{G}$. From (37) this may be written in the form Mel $[f, h]$ = Mel $[g, g]$ and so $[f, h] = [g, g]$. Finally, from (36) this is equivalent to (52).

Theorem 5 contains most of the other theorems as special cases. When $p(x) = q(x) = e^{-x}$, it reduces to Theorem 1, and when $M = N = 1$ it reduces to Theorem 4.

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A THEORY OF NORMAL CHAINS

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Introduction. In this paper we deal with a group-theoretic configuration of the following type: G is an additive group (not necessarily abelian) for which an operator system M and a complete lattice ϕ of M admissible subgroups are defined; we call G and M-ϕ group. In §1 we make various definitions and note that analogues of some of the classical theorems of group theory hold.

Our main interest is in the normal chains for an M-ϕ group. We first discuss normal chains in general, and obtain results which hold if the factors of the chain fulfil suitable conditions (§3). In the remainder of the paper these results are applied to three particular types of normal chain and the relation between these chains is discussed.

The first type of chain discussed is the so-called Loewy chain. This type is of especial importance because it is intimately related to the other two types considered. It is shown how the existence of a Loewy chain connecting the group to 0 may be used in place of chain conditions. Furthermore, if such a chain exists for a nilpotent group, then it is actually a central chain.

We have adopted Hirsch's definition of solubility (or rather its analogue for M-ϕ groups) rather than the customary definition. For the chains usually employed do not meet the general requirements needed to apply our theory. On the other hand, the chains introduced by Hirsch do satisfy these requirements, provided that the group possesses a Loewy chain connecting it to 0.

1. Definitions and basic theorems. Let G be an additive group which is not necessarily abelian. If A_α, for each α in a set $\mathfrak{A}$, is a subgroup of G, then we denote the intersection of the A_α by $\bigcap A_\alpha$ ($\alpha \in \mathfrak{A}$). The subgroup of G generated by the A_α we call the compositum of the A_α and denote this subgroup by $\mathsf{C}\, A_\alpha(\alpha \in \mathfrak{A})$. In the case of a finite number of subgroups, $A_1, \ldots, A_n$, we denote the intersection and compositum by

$$A_1 \cap \ldots \cap A_n \text{ (or } \bigcap_{i=1}^{n} A_i) \quad \text{and} \quad \{A_1, \ldots, A_n\} \text{ (or } \mathsf{C}_{i=1}^{n} A_i)$$

respectively.

Let M be a system of operators for G, so that each element of M induces an endomorphism in G, i.e., we have:

(i) αg is in G, whenever α is in M and g is in G.

(ii) $\alpha(g_1 \pm g_2) = \alpha g_1 \pm \alpha g_2$, for α in M and g_1, g_2 in G.

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We let $(\alpha\beta)g = \alpha(\beta g)$, for α and β in M and g in G.

A subgroup S of G is called M admissible if $MS \subseteq S$. We shall restrict our attention to a family of M admissible subgroups ϕ which form a complete lattice relative to intersection and compositum, i.e., we assume about the subgroups of ϕ that:

(i) If A is in ϕ, A is M admissible.

(ii) 0 and G are in ϕ.

(iii) If A_α is in ϕ, for each α in the set $\mathfrak{A}$, then $\bigcap A_\alpha(\alpha \in \mathfrak{A})$ is in ϕ.

(iv) If A_α is in ϕ, for each α in the set $\mathfrak{A}$, then $\complement A_\alpha(\alpha \in \mathfrak{A})$ is in ϕ.

We note that if the subgroups of ϕ satisfy the descending chain condition, (iii) may be replaced by (iii′), and if they satisfy the ascending chain condition, (iv) may be replaced by (iv′), where:

(iii′) If A and B are in ϕ, then $A \cap B$ is in ϕ.

(iv′) If A and B are in ϕ, then $\{A, B\}$ is in ϕ.

We call G an M-ϕ group if a particular system of operators M, and a particular complete lattice ϕ of M subgroups are to be distinguished; if ϕ consists of all M admissible subgroups we call G an M group. If a subgroup S belongs to ϕ, we say that S is a ϕ subgroup of G; we note that S is also an M-ϕ group. If G is an M-ϕ group, we denote by ψ the set of all normal ϕ subgroups of G; since the ϕ subgroups of G form a complete lattice, the normal ϕ subgroups of G also form a complete lattice. Hence we may also consider G as an M-ψ group. We make the following definitions:

Definition. If the ϕ subgroup S of G has no normal ϕ subgroups, S is ϕ *simple*.

Definition. Let G and G' be M-ϕ groups. σ is an M-ϕ *isomorphism* (*homomorphism*) of G onto G' if

(i) σ is an isomorphism (homomorphism) of G onto G'. (Hence $G' = G\sigma$).

(ii) $(\alpha g)\sigma = \alpha(g\sigma)$, for all α in M and for all g in G.

(iii) If S is a ϕ subgroup of G, $S\sigma$ is a ϕ subgroup of G'; if S' is a ϕ subgroup of G', the inverse image of S', $S'\sigma^{-1}$ is a ϕ subgroup of G. We say that G is M-ϕ *isomorphic* to G' if there exists an M-ϕ isomorphism of G onto G', and we write $G \cong G'(M\text{-}\phi)$.

Definition. Let G and G' be M-ϕ groups. σ is an M-ϕ *isomorphism* (*homomorphism*) of G into G' if $G\sigma \subseteq G'$ and σ is an M-ϕ isomorphism (homomorphism) of G onto $G\sigma$.

In the last two definitions, the systems of distinguished M admissible subgroups for the groups G and G' are both denoted by ϕ, although in general they are different systems. At first sight this would seem to lead to confusion, but it is always clear from the context what is meant and the notation proves to be a convenient one.

Let G be an M-ϕ group, and N a normal ϕ subgroup. Then G/N is an M group, and a system of M admissible subgroups ϕ in G/N may be defined in this way: if U/N is an M admissible subgroup of G/N and U is in ϕ, then U/N is in ϕ. It is clear that this system of subgroups of G/N forms a complete lattice and hence G/N is an M-ϕ group.

The following analogues to the classical theorems hold:

THEOREM 1.1 (The Homomorphism Theorem). *If σ is an M-ϕ homomorphism of the M-ϕ group G onto the M-ϕ group G', then the kernel N is a normal ϕ subgroup of G and G/N is M-ϕ isomorphic to G'. Conversely, if N is a normal ϕ subgroup of the M-ϕ group G, then there exists an M-ϕ homomorphism τ of G onto G/N; τ maps g onto the coset $N + g$, for all g in G, and is called the natural mapping of G onto G/N.*

THEOREM 1.2 (The First Isomorphism Theorem). *If S and T are ϕ subgroups of the M-ϕ group G, and if S is normal in $\{S, T\}$, then $S \cap T$ is a normal ϕ subgroup of T and*

$$\{S, T\}/S \cong T/S\cap T \qquad (M\text{-}\phi).$$

THEOREM 1.3 (The Second Isomorphism Theorem). *Let G be an M-ϕ group and N, H normal ϕ subgroups of G with $N \subseteq H$, then we have:*

$$\frac{G/N}{H/N} \cong \frac{G}{H} \qquad (M\text{-}\phi).$$

Definition. Let A and B be ϕ subgroups of the M-ϕ group G with $A \subseteq B$. If there exists a chain

$$(0) \qquad A = A_0 \subseteq \ldots \subseteq A_i \subseteq A_{i+1} \subseteq \ldots \subseteq A_n = B,$$

where A_i is a normal ϕ subgroup of A_{i+1} $(i = 1, \ldots, n-1)$, (0) is called a *normal* ϕ chain from A to B, or a normal ϕ chain *connecting* A and B. If $A_i \neq A_{i+1}$ for each i, (0) has *length n;* the M-ϕ groups A_{i+1}/A_i are called the *factors* of (0). If all the factors of (0) are ϕ simple, (0) is called a ϕ *composition* chain.

Definition. Let G be an M-ϕ group. A normal ϕ chain (ϕ composition chain) connecting 0 and G is called a normal ϕ chain *for* G (ϕ composition *series*).

Definition. Let G be an M-ϕ group. The ϕ subgroup S of G is M-ϕ *characteristic* if every M-ϕ automorphism (M-ϕ isomorphism of G onto itself) leaves S invariant, i.e., if $S\sigma = S$ for every M-ϕ automorphism σ of G. S is M-ϕ *fully invariant* if every M-ϕ endomorphism of G (M-ϕ homomorphism of G into itself) leaves S invariant, i.e., if $S\tau \subseteq S$ for every M-ϕ endomorphism τ of G.

It is important to notice that the inner automorphisms of a group are not necessarily M-ϕ automorphisms and hence a ϕ subgroup may be M-ϕ characteristic without being normal. In some of our arguments we consider the map of a ϕ subgroup under an inner automorphism. Thus in some cases we make the assumption that ϕ contains conjugates, i.e., if S is in ϕ and g is any element of G, then $-g + S + g$ is in ϕ. If ϕ contains conjugates, we say that ϕ is normal.

2. K-chains. Let (K) be a property which has meaning for each ϕ subgroup of an M-ϕ group, i.e., if S is a ϕ subgroup of the M-ϕ group G, then one of the following statements must be true; S satisfies (K) in G; S does not satisfy (K)

in G. We shall consider properties (K) which satisfy some or all of the following conditions:

(k_1) *If G is an M-ϕ group, then the ϕ subgroup* 0 *satisfies* (K) *in G.*

(k_2) *If for each α in a set $\mathfrak{A}$, A_α is a normal ϕ subgroup of the M-ϕ group G which satisfies* (K) *in G, then $\bigcup A_\alpha (\alpha \in \mathfrak{A})$ satisfies* (K) *in G.*

(k_3) *If A and A_α, for each α in the set $\mathfrak{A}$, are normal ϕ subgroups of the M-ϕ group G with $A \supset A_\alpha$, and if A/A_α satisfies* (K) *in G/A_α, for each α in $\mathfrak{A}$, then $A/\bigcap A_\alpha(\alpha \in \mathfrak{A})$ satisfies* (K) *in $G/\bigcap A_\alpha(\alpha \in \mathfrak{A})$.*

(k_4) *If A, B, C are normal ϕ subgroups of the M-ϕ group G with $A \supset B$, and if A/B satisfies* (K) *in G/B, then $A \cap C/B \cap C$ satisfies* (K) *in $G/B \cap C$.*

(k_5) *If A, B, C are normal ϕ subgroups of the M-ϕ group G with $A \supset B$, and if A/B satisfies* (K) *in G/B, then $\{A, C\}/\{B, C\}$ satisfies* (K) *in $G/\{B, C\}$.*

Let G be an M-ϕ group and ψ the lattice of normal ϕ subgroups of G. If N is in ψ, G/N is an M-ϕ group and hence (K) is defined not only for the ϕ subgroups of G (in G) but for the ϕ subgroups of G/N (in G/N).

We now consider two chains. We construct first the ascending chain:

$$(1) \qquad 0 = T_0 \subseteq T_1 \subseteq \dots \subseteq T_i \subseteq T_{i+1} \subseteq \dots,$$

where, for $i = 0, 1, 2, \dots$, T_{i+1} is the compositum of all N in ψ such that $N \supseteq T_i$ and N/T_i satisfies (K) in G/T_i. Then T_{i+1}/T_i satisfies (K) in G/T_i by (k_2). T_{i+1} is well defined, since by (k_1), T_i/T_i satisfies (K) in G/T_i. We note that in order to construct the chain (1), we need only use the properties (k_1) and (k_2) of (K). Similarly, we construct the descending chain:

$$(2) \qquad G = S_0 \supseteq S_1 \supseteq \dots \supseteq S_j \supseteq S_{j+1} \supseteq \dots,$$

where, for $j = 0, 1, 2, \dots$, S_{j+1} is the intersection of all N in ψ such that $N \subseteq S_j$ and S_j/N satisfies (K) in G/N. Then S_j/S_{j+1} satisfies (K) in G/S_{j+1} by (k_3). S_{j+1} is well defined, since by (k_1), S_j/S_j satisfies (K) in G/S_j. For the construction of the chain (2) only (k_1) and (k_3) are used.

THEOREM 2.1. *Let G be an M-ϕ group.*

(i) *Assume that the property* (K) *satisfies* (k_1) *and* (k_2). *If the ψ subgroups of G satisfy the ascending chain condition, and if for A in ψ, $A \neq G$, there exists a subgroup B in ψ such that $B \supset A$ and B/A satisfies* (K) *in G/A, the chain* (1) *is finite and $T_t = G$ for some integer t.*

(ii) *Assume that the property* (K) *satisfies* (k_1) *and* (k_3). *If the ψ subgroups of G satisfy the descending chain condition, and if for B in ψ, $B \neq 0$, there exists a subgroup A in ψ such that $A \subset B$ and B/A satisfies* (K) *in G/A, the chain* (2) *is finite and $S_s = 0$ for some integer s.*

Proof. (i) The groups T_i of (1) are ψ subgroups of G by definition. Hence by the ascending chain condition, there exists an integer t such that $T_t = T_{t+1}$. If T_t is different from G, there exists a ψ subgroup N of G such that N/T_t satisfies (K) in G/T_t and $N \supset T_t$; but this is impossible, since then T_{t+1} would be different from T_t. Hence $T_t = G$. (ii) is established in a similar fashion.

Definition. Let G be an M-ϕ group. A chain

$$N_0 \subseteq N_1 \subseteq \ldots \subseteq N_i \subseteq N_{i+1} \subseteq \ldots, \tag{3}$$

where, for $i = 0, 1, \ldots, N_i$ is in ψ, and N_{i+1}/N_i satisfies (K) in G/N_i, is called a K-*chain for* G (an *ascending* K-*chain*). A chain

$$M_0 \supseteq M_1 \supseteq \ldots \supseteq M_j \supseteq M_{j+1} \supseteq \ldots, \tag{4}$$

where, for $j = 0, 1, \ldots, M_j$ is in ψ, and M_j/M_{j+1} satisfies (K) in G/M_{j+1}, will also be called a K-*chain for* G (a *descending* K-*chain*). The K-chain

$$N_0 \subset N_1 \subset \ldots \subset N_i \subset \ldots \subset N_n \tag{5}$$

has length n, if for $i = 0, \ldots, n-1$, $N_i \neq N_{i+1}$. The chains (1) and (2) are called the *upper* and *lower* K-chains for G.

THEOREM 2.2. *Let G be an M-ϕ group.*

(i) *Assume that* (K) *satisfies* (k_1), (k_2), *and* (k_5). *If*

$$0 = N_0 \subseteq \ldots \subseteq N_i \subseteq N_{i+1} \subseteq \ldots$$

is an ascending K-*chain for G, then, for $i = 0, 1, \ldots, N_i \subseteq T_i$, where the T_i are the terms of the upper* K-*chain* (1).

(ii) *Assume that* (K) *satisfies* (k_1), (k_3), *and* (k_4). *If*

$$G = M_0 \supseteq \ldots \supseteq M_j \supseteq M_{j+1} \supseteq \ldots$$

is a descending K-*chain for G, then, for $j = 0, 1, \ldots, M_j \supseteq S_j$, where the S_j are the terms of the lower* K-*chain* (2).

Proof. (i). We prove by induction on i that $N_i \subseteq T_i$ for $i = 0, 1, \ldots$. Since $0 = N_0 = T_0$, it is obvious that $N_0 \subseteq T_0$. Assume that $N_i \subseteq T_i$. N_{i+1}/N_i satisfies (K) in G/N_i; therefore, by (k_5),

$$\{N_{i+1}, T_i\}/\{N_i, T_i\} = \{N_{i+1}, T_i\}/T_i$$

satisfies (K) in G/T_i. Hence by the definition of T_{i+1}, $\{N_{i+1}, T_i\} \subseteq T_{i+1}$, or $N_{i+1} \subseteq T_{i+1}$. Thus $N_i \subseteq T_i$ for $i = 0, 1, \ldots$.

(ii). We prove by induction on j that $M_j \supseteq S_j$ for $j = 0, 1, \ldots$. $G = M_0 = S_0$; hence $M_0 \supseteq S_0$. Assume that $M_j \supseteq S_j$. M_j/M_{j+1} satisfies (K) in G/M_{j+1}; therefore, by (k_4), $M_j \cap S_j/M_{j+1} \cap S_j$ satisfies (K) in $G/M_{j+1} \cap S_j$. But $M_j \cap S_j = S_j$ by the induction assumption. Hence by the definition of S_{j+1},

$$S_{j+1} \subseteq M_{j+1} \cap S_j \subseteq M_{j+1}.$$

Thus $S_j \subseteq M_j$ for $j = 0, 1, \ldots$.

COROLLARY 2.1. *Let G be an M-ϕ group and assume that* (k_1)-(k_5) *hold for the property* (K). *Then if there exists a (finite)* K-*chain which connects* 0 *and G, the upper and lower* K-*chains are* K-*chains of shortest length connecting* 0 *and G. If*

$$0 = U_0 \subset \ldots \subset U_i \subset \ldots \subset U_n = G \qquad (\textit{of length } n)$$

is any K-*chain of shortest length, $S_{n-i} \subseteq U_i \subseteq T_i$, for $i = 1, \ldots, n$.*

THEOREM 2.3. *Assume that* (k_1)-(k_5) *hold for the property* (K). *Let G be an M-ϕ group and assume that G has upper and lower* K*-chains of length n connecting* 0 *and G. Then the chains*

$$(6)\quad 0 = T_0 \cap S_{n-1} \subset T_1 \cap S_{n-2} \subset \dots \subset T_i \cap S_{n-i-1} \subset \dots \subset T_{n-1} \cap S_0 = T_{n-1} \subset G$$

and

$$(7)\quad 0 \subset \{T_0, S_{n-1}\} = S_{n-1} \subset \dots \subset \{T_i, S_{n-i-1}\} \subset \dots \subset \{T_{n-1}, S_0\} = G$$

are K*-chains for G.*

Proof. To show that (6) is a K-chain, we have to verify that $T_{i+1} \cap S_{n-i-2} / T_i \cap S_{n-i-1}$ satisfies (K) in $G/T_i \cap S_{n-i-1}$ for $i = 0, \dots, n-2$. By the definition of T_i, T_{i+1}/T_i satisfies (K) in G/T_i for $i = 0, \dots, n-1$. Therefore, by (k_4),

$$T_{i+1} \cap S_{n-i-2}/T_i \cap S_{n-i-2}$$

satisfies (K) in $G/T_i \cap S_{n-i-2}$. By the definition of S_j, S_{n-i-2}/S_{n-i-1} satisfies (K) in G/S_{n-i-1} for $i = 0, \dots, n-2$. Therefore, by (k_4),

$$T_{i+1} \cap S_{n-i-2}/T_{i+1} \cap S_{n-i-1}$$

satisfies (K) in $G/T_{i+1} \cap S_{n-i-1}$. Hence, by (k_3),

$$T_{i+1} \cap S_{n-i-2}/T_i \cap S_{n-i-2} \cap T_{i+1} \cap S_{n-i-1} = T_{i+1} \cap S_{n-i-2}/T_i \cap S_{n-i-1}$$

satisfies (K) in $G/T_i \cap S_{n-i-1}$, and (6) is a K-chain for G.

To show that (7) is a K-chain, we have to verify that $\{T_{i+1}, S_{n-i-2}\}/T_i, S_{n-i-1}\}$ satisfies (K) in $G/\{T_i, S_{n-i-1}\}$ for $i = 0, \dots, n-2$. Since T_{i+1}/T_i satisfies (K) in G/T_i, we deduce from (k_5) that

$$\{T_{i+1}, S_{n-i-1}\}/\{T_i, S_{n-i-1}\}$$

satisfies (K) in $G/\{T_i, S_{n-i-1}\}$. Also since S_{n-i-2}/S_{n-i-1} satisfies (K) in G/S_{n-i-1}, $\{T_i, S_{n-i-2}\}/\{T_i, S_{n-i-1}\}$ satisfies (K) in $G/\{T_i, S_{n-i-1}\}$. Hence by (k_2),

$$\{\{T_{i+1}, S_{n-i-1}\}, \{T_i, S_{n-i-2}\}\}/\{T_i, S_{n-i-1}\} = \{T_{i+1}, S_{n-i-2}\}\{/T_i, S_{n-i-1}\}$$

satisfies (K) in $G/\{T_i, S_{n-i-1}\}$, and (7) is a K-chain for G.

The K-chains (1), (2), (6) and (7) are shown in the accompanying Hasse diagram.

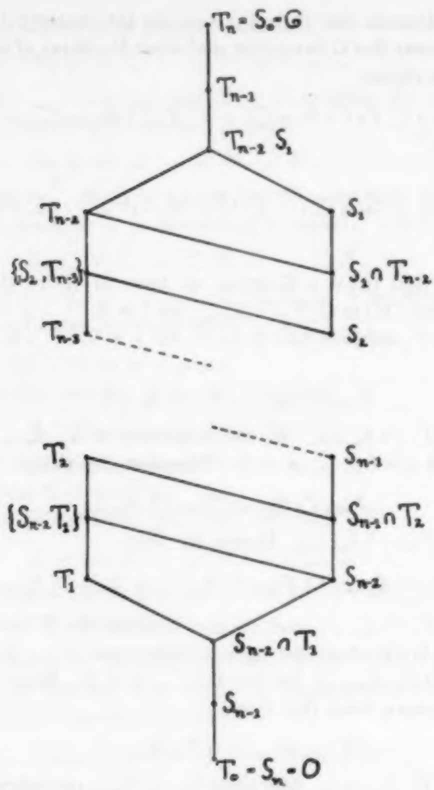
THEOREM 2.4. *Let* (K) *be a property for which* (k_4) *holds. If the M-ϕ group G has a* K*-chain connecting* 0 *and G, let*

$$(8)\qquad 0 = U_0 \subset \dots \subset U_i \subset U_{i+1} \subset \dots \subset U_n = G,$$

and

$$(9)\qquad 0 = V_0 \subset \dots \subset V_i \subset V_{i+1} \subset \dots \subset V_n = G$$

be K*-chains of shortest length. Then* U_{i+1} not $\subseteq V_i$, *for* $i = 0, \dots, n-1$.



Proof. Suppose that $U_{i+1} \subseteq V_i$; then

$$\begin{aligned} 0 = V_0 \cap U_{i+1} \subseteq \ldots \subseteq V_j \cap U_{i+1} \subseteq \ldots \subseteq V_i \cap U_{i+1} \\ = U_{i+1} \subset U_{i+2} \subset \ldots \subset U_n = G \end{aligned}$$

is a K-chain, and its length is less than n. But this is impossible and hence U_{i+1} not $\subseteq V_i$.

Consequently if G has a K-chain connecting 0 and G we have for the upper and lower K-chains, provided that (K) satisfies (k_1)-(k_5):

$$\text{(i)} \quad S_{n-i-1} \text{ not} \subseteq T_i \qquad (i = 0, \ldots, n-1),$$
$$\text{(ii)} \quad T_{i+1} \text{ not} \subseteq S_{n-i} \qquad (i = 0, \ldots, n-1).$$

The properties (K) which we shall discuss are also invariant under M-ϕ isomorphisms of the group. That is:

(k_6) *Let G be an M-ϕ group and σ an M-ϕ isomorphism of G. Then if the normal ϕ subgroup A satisfies* (K) *in G, $A\sigma$ satisfies* (K) *in $G\sigma$.*

The conditions (k_5) and (k_6) are equivalent to the following condition:

(k'_5) *Let G be an M-ϕ group and η an M-ϕ homomorphism of G. If A and B are normal ϕ subgroups with $A \supset B$, and if A/B satisfies* (K) *in G/B, then $A\eta/B\eta$ satisfies* (K) *in $G\eta/B\eta$.*

Proof. Clearly (k'_5) implies (k_6). We show next that (k'_5) also implies (k_5). We assume that A and B are normal ϕ subgroups of the M-ϕ group G with $A \supset B$ and that A/B satisfies (K) in G/B. If C is a normal ϕ subgroup of G, let η be the natural mapping of G onto G/C. Then $A\eta = \{A, C\}/C$ and $B\eta = \{B,C\}/C$. Thus by (k'_5),

$$\frac{\{A, C\}/C}{\{B, C\}/C} \text{ satisfies (K) in } \frac{G/C}{\{B, C\}/C}.$$

But (K) is invariant under M-ϕ isomorphism and hence $\{A, C\}/\{B, C\}$ satisfies (K) in $G/\{B,C\}$.

Conversely, we show that (k_5) and (k_6) imply (k'_5). Assume that A and B are normal ϕ subgroups of the M-ϕ group G with $A \supset B$ and that A/B satisfies (K) in G/B. Let η be an M-ϕ homomorphism of G, and let C be the kernel of η. Then C is a normal ϕ subgroup of G and the natural homomorphism of G onto G/C takes A onto $\{A, C\}/C$ and B onto $\{B, C\}/C$. Hence by the Homomorphism Theorem there exists an M-ϕ isomorphism of $G\eta$ onto G/C which takes $A\eta$ onto $\{A, C\}/C$ and B onto $\{B, C\}/C$.

By the Second Isomorphism Theorem there exists an M-ϕ isomorphism of

$$\frac{G/C}{\{B, C\}/C} \text{ onto } G/\{B, C\}$$

which takes

$$\frac{\{A, C\}/C}{\{B, C\}/C} \text{ onto } \{A, C\}/\{B, C\}.$$

Therefore, there exists an M-ϕ isomorphism σ of $G/\{B, C\}$ onto $G\eta/B\eta$ with

$$A\eta/B\eta = (\{A, C\}/\{B, C\})\sigma.$$

By (k_5), $\{A, C\}/\{B, C\}$ satisfies (K) in $G/\{B, C\}$, and from (k_6) it follows that $A\eta/B\eta$ satisfies (K) in $G\eta/B\eta$.

THEOREM 2.5. *Let G be an M-ϕ group and* (K) *a property for which* (k_1)-(k_6) *hold. The terms of the upper and lower* K*-chains are M-ϕ characteristic.*

Proof. We prove by induction that the terms of the upper K-chain are M-ϕ characteristic. $T_0 = 0$ and hence is M-ϕ characteristic. Assume that T_i is M-ϕ characteristic; and let η be an M-ϕ automorphism of G. Then η induces an M-ϕ automorphism $\bar{\eta}$ of G/T_i, since $T_i\eta = T_i$. We deduce from (k_6) that

$$T_{i+1}\eta/T_i = (T_{i+1}/T_i)\bar{\eta}$$

satisfies (K) in $G/T_i = (G/T_i)\bar{\eta}$. Hence by the definition of T_{i+1}, $T_{i+1}\eta \subseteq T_{i+1}$. Similarly, $T_{i+1}\eta^{-1} \subseteq T_{i+1}$ so that $T_{i+1}\eta = T_{i+1}$.

We prove by induction that the terms of the lower K-chain are M-ϕ characteristic. $S_0 = G$ and hence is M-ϕ characteristic. Assume that S_j is M-ϕ characteristic and let η be an M-ϕ automorphism of G. By (k'_5), $S_j\eta/S_{j+1}\eta$ satisfies(K) in $G\eta/S_{j+1}\eta$ which implies that $S_{j+1} \subseteq S_{j+1}\eta$. Since a similar argument shows that $S_{j+1} \subseteq S_{j+1}\eta^{-1}$, $S_{j+1}\eta = S_{j+1}$.

In this section we have often made the hypothesis that the property (K) satisfies certain ones of the conditions (k_1)-(k_6). It may happen, of course, that (K) satisfies these (k_i) for some M-ϕ groups but not for others. In the following sections we shall often restrict the class of M-ϕ groups considered, and discuss particular properties (K) for this class. It is clear that the results of this section may be applied to this class of groups, provided that (K) satisfies suitable conditions (k_i) for groups in this class, and provided that the ψ subgroups and quotient groups of a group in the class also belong to the class.

3. Loewy chains. The first property (K) which we shall consider gives rise to the so-called Loewy chains [**2**, pp. 506-509]. Following Remak, we make the following definitions:

Definition. Let G be an M-ϕ group. If F is a minimal normal ϕ subgroup ($\neq 0$) of G, we call F a *foot* of G.

Definition. The compositum of all feet of the M-ϕ group G is called the *socle* and is denoted by $S = S(G)$. (If G has no feet, the socle is defined to be 0.)

Before defining Loewy chains we state the following results [**7**]:

LEMMA 3.1. *If T is a normal ϕ subgroup of the M-ϕ group G and if $T = \complement F(\alpha \in \mathfrak{A})$ where F_α is a foot of G for each α of the set $\mathfrak{A}$, then there exists a subset $\mathfrak{B}$ of $\mathfrak{A}$ such that $T = \sum^\circ F_\beta(\beta \in \mathfrak{B})$. (The notation $\sum^\circ$ is used for direct sum.)*

Remak proves this in the case where $\mathfrak{A}$ is finite. The same method of proof is valid in the infinite case, using transfinite induction.

COROLLARY 3.1. *Let G be an M-ϕ group with socle $S \neq 0$. S is the direct sum of feet of G.*

LEMMA 3.2. *If N is a normal ϕ subgroup of the M-ϕ group G contained in the socle S of G, N is the direct sum of feet of G. Furthermore, there exists a normal ϕ subgroup N' of G such that $S = N \oplus N'$.*

Proof. Let K be the compositum of all feet F of G with $F \subseteq N$. By Lemma 3.1, there exist sets $\mathfrak{A}$ and $\mathfrak{B}$ such that

$$K = \sum{}^\circ F_\alpha \ (\alpha \in \mathfrak{U}), \quad S = K \oplus \sum{}^\circ F_\beta \ (\beta \in \mathfrak{B})$$

where F_α and F_β are feet of G for α in $\mathfrak{A}$ and β in $\mathfrak{B}$ respectively. $N \supseteq K$ and hence $N = K \oplus (N \cap \sum^\circ F_\beta(\beta \in \mathfrak{B}))$.

Assume that $N \cap \sum^\circ F_\beta \neq 0$ $(\beta \in \mathfrak{B})$. Let x be a non-zero element of

$N \cap \sum F_\beta(\beta \in \mathfrak{B})$; then $x = f_1 + \ldots + f_n$, where, for $i = 1, \ldots, n$, f_i is in F_{β_i} and β_i is in $\mathfrak{B}$. Hence

$$L = N \cap \sum_{i=1}^{n}{}^{\circ} F_{\beta_i} \neq 0.$$

$\sum_{i=1}^{n}{}^{\circ} F_{\beta_i}$ is a ψ subgroup of G and its ψ subgroups satisfy the minimum condition. Hence there exists a minimal ψ subgroup $F \neq 0$ contained in L. Thus F is a foot of G and is contained in N. But this is impossible because then $F \subseteq K$ and $K \cap L = 0$. Therefore

$$N \cap \sum{}^{\circ} F_\beta = 0 \ (\beta \in \mathfrak{B}), N = K = \sum{}^{\circ} F_\alpha \ (\alpha \in \mathfrak{A}),$$

so that N is the direct sum of feet of G.

Let $N' = \sum^{\circ} F_\beta(\beta \in \mathfrak{B})$; then N' is a normal ϕ subgroup of G and $S = N \oplus N'$.

Corollary 3.2. *If N is normal ϕ subgroup of the M-ϕ group G contained in the socle S of G, then S/N is the direct sum of feet of G/N.*

Proof. By Lemma 3.2, $S = N \oplus N'$ and $N' = \sum^{\circ} F_\beta(\beta \in \mathfrak{B})$, where F_β is a foot of G, for β in $\mathfrak{B}$. S/N is therefore M-ψ isomorphic to N' and hence is the direct sum of feet of G/N.

Consider now the following property of ϕ subgroups of an M-ϕ group:

(R). Let A be a ϕ subgroup of the M-ϕ group G. A satisfies (R) in G if A is contained in the socle of G.

Definition. A normal ϕ subgroup N of the M-ϕ group G is *fully reducible* with respect to G if it is the compositum of feet of G. (We assume that 0, which is the sum ot no feet, is fully reducible.)

From Lemma 3.2 we see that a normal ϕ subgroup satisfies (R) in G if and only if it is fully reducible with respect to G.

We call an R-chain a Loewy chain. The property (R) obviously satisfies (k_1) and (k_2) so that the upper Loewy chain may be constructed. We denote the upper Loewy chain by:

$$(10) \qquad 0 = S_0 \subseteq \ldots \subseteq S_i \subseteq S_{i+1} \subseteq \ldots.$$

We verify that (R) satisfies (k_3):

Theorem 3.1. *Let A, B, and C be normal ϕ subgroups of the M-ϕ group G with $A \supset B$. If A/B is fully reducible with respect to G/B, then $\{A, C\}/\{B, C\}$ is fully reducible with respect to $G/\{B,C\}$.*

Proof. Since A/B is fully reducible with respect to G/B, $A/B = \complement\,(A_\alpha/B)$ $(\alpha \in \mathfrak{A})$, where A_α/B is a foot of G/B, for each α in the set $\mathfrak{A}$.

$$\frac{\{A, C\}}{\{B, C\}} = \frac{\{\complement A_\alpha, C\}}{\{B, C\}} = \frac{\complement\{A_\alpha, C\}}{\{B, C\}} \qquad (\alpha \in \mathfrak{A}).$$

Now

$$\frac{\{A_s, C\}}{\{B, C\}} = \frac{\{A_s, \{B, C\}\}}{\{B, C\}} \cong \frac{A_s}{A_s \cap \{B, C\}} \qquad (M\text{-}\psi);$$

A_s is a minimal ψ subgroup of G which contains B. Hence since $A_s \cap \{B, C\}$ is a ψ subgroup of G and $B \subseteq A_s \cap \{B, C\} \subseteq A_s$, either

$$B = A_s \cap \{B, C\} \text{ or } A_s = A_s \cap \{B, C\} \text{ so that } A_s \subseteq \{B, C\}.$$

In the first case,

$$\frac{\{A_s, C\}}{\{B, C\}} \cong \frac{A_s}{B} \qquad (M\text{-}\psi)$$

and hence $\{A_s, C\}/\{B, C\}$ is a foot of $G/\{B, C\}$. In the second case, $\{A_s, C\} = \{B, C\}$. Therefore, $\{A, C\}/\{B, C\}$ is fully reducible with respect to $G/\{B, C\}$.

Thus the condition (R) satisfies (k_1), (k_2), and (k_3) and hence as a consequence of Theorem 2.2 we have:

THEOREM 3.2. *Let G be an M-ϕ group which possesses a Loewy chain*

$$0 = N_0 \subseteq \ldots \subseteq N_i \subseteq N_{i+1} \subseteq \ldots$$

(that is, N_i is normal in G, and N_{i+1}/N_i is fully reducible with respect to G/N_i, for $i = 0, 1, \ldots$), then $N_i \subseteq S_i$, where the S_i are the terms of the upper Loewy chain. Hence if $N_n = G$ for some integer n, $S_n = G$ so that the upper Loewy chain connects 0 and G and has length $\leqslant n$.

If J is a maximal ψ subgroup of the M-ϕ group G, then G/J is ϕ simple and hence is fully reducible with respect to G/J. Hence if (k_3) were satisfied by (R) we should have G/N fully reducible (with respect to G/N) for N the intersection of maximal ψ subgroups of G. That this is not in general the case is shown by a simple example:

EXAMPLE 3.21. Let G be the additive group of integers, M void and let ϕ consist of all subgroups of G. Then if p is any prime, (p), the group generated by p, is a maximal normal subgroup of G. Furthermore,

$$\bigcap_{i=1}^{\infty}(p_i) = 0$$

if $p_1, p_2, \ldots$ is an infinite sequence of different primes. But G contains no minimal subgroups, and hence is certainly not fully reducible; in fact, the upper Loewy chain has as its only term 0.

We shall need the following theorem to show that (k_3) holds for the property (R) in an M-ϕ group G, provided that the ψ subgroups of G satisfy the minimum condition:

THEOREM 3.3. *Let A and B be normal ϕ subgroups of the M-ϕ group G with $A \supset B$. If A/B is fully reducible with respect to G/B, then B is the intersection of maximal ψ subgroups of A.*

Proof. (i) Assume that $B = 0$. By Lemma 3.1, there exists a set $\mathfrak{B}$ such that $A = \Sigma^\circ A_\beta(\beta \in \mathfrak{B})$ where, for β in $\mathfrak{B}$, A_β is a foot of G. For δ in $\mathfrak{B}$ define

$$J_\delta = \sum_{\beta \neq \delta} A_\beta \ (\beta \in \mathfrak{B}).$$

Then $A = J_\delta \oplus A_\delta$, and J_δ is a maximal ψ subgroup of A.

Let $K = \bigcap J_\delta$; K is a ψ subgroup of A and hence if $K \neq 0$, $K = \Sigma A_\gamma(\gamma \in \mathfrak{C})$, where $\mathfrak{C}$ is non-void, and $\mathfrak{C} \subset \mathfrak{B}$. If γ is in $\mathfrak{C}$, $A_\gamma \subseteq \bigcap J_\delta = K$ $(\delta \in \mathfrak{B})$ so that in particular $A_\gamma \subseteq J_\gamma$. But this is impossible. Therefore, $K = \bigcap J_\delta = 0$, and $B = 0$ is the intersection of maximal ψ subgroups of A.

(ii) In the general case if we apply the result of (i) to the quotient group, G/B, we have: There exists a set $\mathfrak{B}$ such that $B/B = \bigcap (J_\gamma/B)$ $(\gamma \in \mathfrak{B})$, where J_γ/B is a maximal ψ subgroup of A/B, for γ in $\mathfrak{B}$. But then

$$B = \bigcap_\gamma J_\gamma,$$

and J_γ is a maximal ψ subgroup of A, for γ in $\mathfrak{B}$.

THEOREM 3.4. *Assume that the ψ subgroups of the M-ϕ group G satisfy the minimum condition. Let A, A_α be ψ subgroups of G for each α in the set $\mathfrak{A}$, with $A_\alpha \subset A$. Then if A/A_α is fully reducible with respect to G/A_α, for α in $\mathfrak{A}$, $A/\bigcap A_\alpha$ is fully reducible with respect to $G/\bigcap A_\alpha$.*

Proof. By Theorem 3.3, A_α is the intersection of maximal ψ subgroups of A. Hence $\bigcap A_\alpha$ is the intersection of maximal ψ subgroups of A, and is therefore the intersection of a finite number of maximal ψ subgroups of A, since the ψ subgroups of G satisfy the minimum condition.

Let

$$C = \bigcap A_\alpha = \bigcap_{i=1}^{n} M_i,$$

where M_i $(i = 1, \ldots, n)$ is a maximal ψ subgroup of A, and assume that $n > 1$ and that

$$K_j = \bigcap_{i(\neq j)=1}^{n} M_i \neq C \qquad (j = 1, \ldots, n).$$

Then since M_1 is a maximal ψ subgroup of A and K_1 is not contained in M_1, $A = \{K_1, M_1\}$. M_1 has the maximal ψ subgroups

$$M_1 \cap M_2, \ \ldots, \ M_1 \cap M_n; \text{ and } K_2 = \bigcap_{i=3}^{n} (M_1 \cap M_i)$$

so that the same argument applied to M_1 shows that $M_1 = \{K_2, M_1 \cap M_2\}$. Continuing in this manner we obtain

$$A = \mathop{\mathsf{C}}_{j=1}^{n} K_j.$$

Hence

$$A/C = \mathop{\mathrm{C}}_{j=1}^{n} (K_j/C),$$

$$\frac{K_j}{C} = \frac{K_j}{K_j \cap M_j} \cong \frac{\{M_j, K_j\}}{M_j} \qquad (M\text{-}\psi)$$
$$= A/M_j,$$

which is ψ simple. Thus K_j/C is a foot of G/C, so that $A/\bigcap A_\alpha = A/C$ is fully reducible with respect to G/C.

Hence (R) satisfies (k_3) for M-ϕ groups whose ψ subgroups satisfy the minimum condition, and the lower Loewy chain may be constructed for these groups. We denote the lower Loewy chain by:

$$G = R_0 \supseteq \ldots \supseteq R_j \supseteq R_{j+1} \supseteq \ldots . \tag{11}$$

THEOREM 3.5. *Assume that the ψ subgroups of the M-ϕ group G satisfy the minimum condition. Let A and B be ψ subgroups of G with $B \subset A$. If A/B is fully reducible with respect to G/B, then $A \cap C/B \cap C$ is fully reducible with respect to $G/B \cap C$.*

Proof. By Theorem 3.3,

$$B = \bigcap_{i=1}^{n} M_i,$$

where M_i $(i = 1, \ldots, n)$ is a maximal ψ subgroup of A. Hence

$$B \cap C = \bigcap_{i=1}^{n} (M_i \cap C).$$

If $A \cap C \neq M_i \cap C$,

$$\frac{A \cap C}{M_i \cap C} = \frac{A \cap C}{M_i \cap (A \cap C)} \cong \frac{\{M_i, A \cap C\}}{M_j} \qquad (M\text{-}\psi),$$

which is ψ simple. Hence $M_i \cap C$ is a maximal ψ subgroup of $A \cap C$; and Theorem 3.4 shows that $A \cap C/B \cap C$ is fully reducible with respect to $G/B \cap C$.

COROLLARY 3.3. *If the ψ subgroups of the M-ϕ group G satisfy the minimum condition, then for G the property* (R) *satisfies the conditions* (k_1)-(k_6).

Hence we have:

THEOREM 3.6. *Let G be an M-ϕ group whose ψ subgroups satisfy the minimum condition, and assume that G possesses a Loewy chain:*

$$G = K_0 \supseteq \ldots \supseteq K_j \supseteq K_{j+1} \supseteq \ldots$$

(that is, K_j is in ψ, and K_j/K_{j+1} is fully reducible with respect to G/K_{j+1}, for $j = 0, 1, \ldots$) Then $K_j \supseteq R_j$ for $j = 0, 1, \ldots$, where the R_j are the terms of the lower Loewy chain (11). *Hence if $K_n = 0$ for some integer n, $R_n = 0$ so that the lower Loewy chain connects G and 0 and has length $\leqslant n$.*

COROLLARY 3.4. *Under the hypotheses of the preceding theorem (that is, the ψ subgroups of G satisfy the minimum condition and $K_n = 0$ for some integer n), the upper and lower Loewy chains connect 0 and G and have equal lengths.*

THEOREM 3.7. *Let G be an M-ϕ group and assume that the upper Loewy chain connects 0 and G so that*

$$0 = S_0 \subset \ldots \subset S_i \subset \ldots \subset S_n = G.$$

Then if we define the chain

$$G = M_0 \supseteq \ldots \supseteq M_j \supseteq M_{j+1} \supseteq \ldots,$$

where M_{j+1} is the intersection of M_j with all maximal ψ subgroups of M_j, there exists an integer $m \leqslant n$ such that $M_m = 0$.

Proof. We use induction to prove that $M_i \subseteq S_{n-i}$. Since S_{n-1} is the intersection of maximal ψ subgroups of G, $M_1 \subseteq S_{n-1}$. Assume that $M_j \subset S_{n-j}$; by Theorem 3.3, S_{n-j-1} is the intersection of maximal ψ subgroups of S_{n-j} so that there exists a set $\mathfrak{A}$ such that N_a is a maximal ψ subgroup of S_{n-j} for a in $\mathfrak{A}$, and

$$S_{n-j-1} = \bigcap N_a \qquad (a \in \mathfrak{A}).$$

Either $\{M_j, N_a\} = N_a$ or S_{n-j}. In the first case, $M_j \cap N_a = M_j$; in the second, $M_j \cap N_a$ is a maximal ψ subgroup of M_j, since

$$\frac{M_j}{M_j \cap N_a} \cong \frac{\{M_j, N_a\}}{M_j} = \frac{S_{n-j}}{M_j} \qquad (M\text{-}\psi)$$

which is ψ simple. Thus

$$\bigcap_a (M_j \cap N_a)$$

is the intersection with M_j of maximal ψ subgroups of M_j so that

$$M_{j+1} \subseteq \bigcap_a (M_j \cap N_a) \subseteq \bigcap_a N_a = S_{n-j-1}.$$

Hence $M_i \subseteq S_{n-i}$ $(i = 0, 1, \ldots, n)$. In particular, $M_n \subseteq S_0 = 0$ and $M_n = 0$.

As we have seen the converse of Theorem 3.7 is not true, for Example 3.21 shows that even if $M_1 = 0$, there may be no Loewy chain connecting 0 and G. Under the hypothesis of Theorem 3.7, it is not possible to prove that if $M_m = 0$, $S_m = G$, as the following example shows:

EXAMPLE 3.71. Let W be the direct sum of the cyclic groups generated by b, $b_1, \ldots, b_i, \ldots$, elements of prime order p; thus

$$W = (b) \oplus (b_1) \oplus \ldots \oplus (b_i) \oplus \ldots.$$

Let M consist of the endomorphisms ρ_i, where $\rho_i(b) = b_i$, $\rho_i(b_i) = b_i$ and $\rho_i(b_j) = 0$ $(j \neq i)$. Let ϕ consist of all M admissible subgroups of W.

Then $V = (b_1) \oplus \ldots \oplus (b_i) \oplus \ldots$ is the socle of W and $0 \subset V \subset W$ is the upper Loewy chain for W. If

$$V_i = (b_1) \oplus \ldots \oplus (b_{i-1}) \oplus (b - b_i) \oplus (b_{i+1}) \oplus \ldots,$$

V_i is a maximal ϕ subgroup of W, and

$$\bigcap_{i=1}^{n} V_i = (b - b_1 - \ldots - b_n) \oplus (b_{n+1}) + \ldots,$$

so that

$$\bigcap_{i=1}^{\infty} V_i = 0.$$

Hence although the length of the shortest Loewy chain for W is 2, the intersection of all maximal (normal) ϕ subgroups is 0.

4. Central chains. The centre of an M-ϕ group G is not necessarily a ϕ subgroup of G. However, if for α in the set $\mathfrak{A}$, S_α is a ϕ subgroup contained in the centre of G, the compositum of the S_α is a ϕ subgroup which is contained in the centre of G.

Definition. Let G be an M-ϕ group. The ϕ *centre* of G is the compositum of all the ϕ subgroups which are contained in the centre of G, and is denoted by $Z_\phi(G)$.

The ϕ centre is the uniquely determined greatest ϕ subgroup of G all of whose elements are centre elements, and is obviously normal in G.

In this section we shall consider Z-chains, or central chains, where the property (Z) is defined by:

(Z) The ϕ subgroup A of the M-ϕ group G satisfies (Z) in G if $A \subseteq Z_\phi(G)$.

Clearly (k_1) and (k_2) hold for (Z).

THEOREM 4.1. *If A and B are normal ϕ subgroups of the M-ϕ group G with $A \supset B$, and if A/B is contained in $Z_\phi(G/B)$, then for any M-ϕ homomorphism η of G, $A\eta/B\eta$ is contained in $Z_\phi(G\eta/B\eta)$. Hence* (k'_5) *holds for* (Z).

Proof. Let a be an element of A, g an element of G; then

$$-a\eta - g\eta + a\eta + g\eta = (-a - g + a + g)\eta,$$

which is in $B\eta$, since $-a - g + a + g$ is in B. Hence $A\eta/B\eta \subseteq Z_\phi(G\eta/B\eta)$.

Definition. Let G be an M-ϕ group. We make the inductive definition:

$$Z_0 = Z_0(G) = 0, \quad Z_{\nu+1}/Z_\nu = Z_{\nu+1}(G)/Z_\nu(G) = Z_\phi(G/Z_\nu)$$

for all ordinals $\nu \geq 0$, and

$$Z_\lambda = Z_\lambda(G) = \bigcup_{\nu<\lambda} Z_\nu(G),$$

for limit ordinals λ.

The groups Z_i, for positive integral i, are the terms of the upper central chain and hence are M-ϕ characteristic by Theorem 2.5; it is easily verified (by transfinite induction) that Z_ν is normal and M-ϕ characteristic, for each ordinal ν.

THEOREM 4.2. *Assume that the M-ϕ group G possesses a central chain,*

$$0 = N_0 \subseteq \ldots \subseteq N_i \subseteq N_{i+1} \subseteq \ldots,$$

then $N_i \subseteq Z_i$, *for* $i = 0, 1, \ldots$. *If* $N_n = G$ *for some integer* n, *the upper central chain is finite of length* $c \leqslant n$, *and connects* 0 *and* G.

Proof. It has been shown that (k'_5) implies (k_5) so that (Z) satisfies (k_1), (k_2), and (k_5). Hence the theorem follows from Theorem 2.2 (i).

Definition. Let A and B be normal ϕ subgroups of the M-ϕ group G. Then (A, B) is the intersection of all normal ϕ subgroups of $\{A, B\}$ which contain $-a-b+a+b$, for all a in A and all b in B.

Thus (A, B) is the smallest normal ϕ subgroup of $\{A, B\}$ which contains all the commutators $-a-b+a+b$.

LEMMA 4.1. *Let* A *and* B *be normal* ϕ *subgroups of the* M-ϕ *group* G *with* $B \subset A$. A/B *is contained in* $Z_\phi(G/B)$ *if and only if* (A, G) *is contained in* B.

Proof. Assume that $A/B \subseteq Z_\phi(G/B)$. If a and g are elements of A and G respectively, $-a-g+a+g$ is an element of B. Thus B is a normal ϕ subgroup of $G = \{A, G\}$ which contains $-a-g+a+g$, for all a in A and all g in G; hence $(A, G) \subseteq B$. Conversely, if $B \supseteq (A, G)$, the element $-a-g+a+g$ is in B, for all a in A and all g in G; hence

$$a + g \equiv g + a \pmod{B},$$

or $A/B \subseteq Z_\phi(G/B)$.

THEOREM 4.3. *Let* G *be an* M-ϕ *group.*

(i) *If* A *and* A_α, *for each* α *in the set* $\mathfrak{A}$, *are normal* ϕ *subgroups with* $A \supset A_\alpha$, *and if* $A/A_\alpha \subseteq Z_\phi(G/A_\alpha)$, *for* α *in* $\mathfrak{A}$, *then*

$$A / \bigcap A_\alpha \subseteq Z_\phi(G/ \bigcap A_\alpha);$$

hence (k_3) *holds for* (Z).

(ii) *If* A, B *and* C *are normal* ϕ *subgroups with* $A \supset B$, *and if* $A/B \subseteq Z_\phi(G/B)$, *then*

$$A \cap C/B \cap C \subseteq Z_\phi(G/B \cap C);$$

hence (k_4) *holds for* (Z).

Proof. (i) By Lemma 4.1, $A_\alpha \supseteq (A, G)$, for α in $\mathfrak{A}$; therefore, $\bigcap A_\alpha \supseteq (A, G)$ so that

$$A/ \bigcap A_\alpha \subseteq Z_\phi(G/ \bigcap A_\alpha).$$

(ii) Since, by Lemma 4.1, $(A, G) \subseteq B$, $(A \cap C, G) \subseteq (A, G) \subseteq B$. Since C is normal in G, the element $-c-g+c+g$ is in C, for all c in C and g in G. Thus $(A \cap C, G) \subseteq C$. Therefore, $(A \cap C, G) \subseteq B \cap C$, and by Lemma 4.1,

$$A \cap C/B \cap C \subseteq Z_\phi(G/B \cap C).$$

Definition. Let G be an M-ϕ group. We define by transfinite induction:

$$C^0(G) = G, \quad C^{\nu+1}(G) = (C^\nu(G), G)$$

for all ordinals $\nu > 0$, and

$$C^\lambda(G) = \bigcap_{\nu<\lambda} C^\nu(G)$$

for limit ordinals λ.

The groups $C^i(G)$, for positive integral i, are the terms of the lower central chain. For by Lemma 4.1, $C^{i+1}(G)$ is the smallest normal ϕ subgroups of G in $C^i(G)$ such that $C^i(G)/C^{i+1}(G)$ is contained in $Z_\phi(G/C^{i+1}(G))$. It is easily verified (by transfinite induction) that $C^\nu(G)$ is M-ϕ fully invariant for each ordinal ν; $C^{\nu+1}(G)$ is normal in G (by definition) and, for λ a limit ordinal, $C^\lambda(G)$ is obviously normal in G.

Lemma 4.2. *If N is a normal ϕ subgroup of the M-ϕ group G, then*

$$C^i(G/N) = \{C^i(G), N\}/N \qquad (i = 0, 1, \ldots).$$

Proof. We use induction on i. The lemma is true for $i = 0$, since

$$C^0(G/N) = G/N = \{G, N\}/N = \{C^0(G), N\}/N.$$

Assume that the lemma is true for $i = j$, that is, assume that

$$C^j(G/N) = \{C^j(G), N\}/N,$$

and let $C^{j+1}(G/N) = K/N$. Then

$$\frac{C^j(G/N)}{C^{j+1}(G/N)} \subseteq Z_\phi\left(\frac{G/N}{C^{j+1}(G/N)}\right)$$

or

$$\frac{\{C^j(G), N\}/N}{K/N} \subseteq Z_\phi\left(\frac{G/N}{K/N}\right);$$

hence $\{C^j(G), N\}/K \subseteq Z_\phi(G/K)$. Thus

$$K \supseteq (\{C^j(G), N\}, G) \supseteq (C^j(G), G) = C^{j+1}(G)$$

so that

$$K \supseteq \{C^{j+1}(G), N\}. \tag{12}$$

On the other hand, since $C^j(G)/C^{j+1}(G)$ is contained in $Z_\phi(G/C^{j+1}(G))$, we deduce from property (k_5) that

$$\frac{\{C^j(G), N\}}{\{C^{j+1}(G), N\}} \subseteq Z_\phi\left(\frac{G}{\{C^{j+1}(G), N\}}\right);$$

hence

$$\frac{\{C^j(G), N\}/N}{\{C^{j+1}(G), N\}/N} \subseteq Z_\phi\left(\frac{G/N}{\{C^{j+1}(G), N\}/N}\right).$$

Thus

$$C^{j+1}(G/N) \subseteq \{C^{j+1}(G), N\}/N, \tag{13}$$

and combining (12) and (13) we obtain $K/N = C^{j+1}(G/N) = \{C^{j+1}(G), N\}/N$. The induction is thus complete, and $C^i(G/N) = \{C^i(G), N\}/N$ $(i = 0, 1, \ldots)$.

Theorem 4.4. *Let G be an M-ϕ group which possesses a central chain*

$$G = M_0 \supseteq \ldots \supseteq M_j \supseteq M_{j+1} \ldots,$$

then $M_j \supseteq C^j(G)$, for $j = 0, 1, \ldots$. If $M_n = 0$ for some integer n, then the lower central chain connects G and 0 and has length $\leqslant n$.

Proof. Since (Z) satisfies (k_1), (k_2), and (k_4), this follows from Theorem 2.2 (ii).

COROLLARY 4.1. *If the M-ϕ group G possesses a central chain of length n connecting 0 and G, the upper and lower central chains are of equal length $c \leqslant n$ and both connect 0 and G.*

Definition. The M-ϕ group G is *ϕ nilpotent of finite class c*, if the upper central chain connects 0 and G and has length c.

THEOREM 4.5. *If the M-ϕ group G is ϕ nilpotent of finite class c, then*
(i) *Any ϕ subgroup S is ϕ nilpotent of finite class $\leqslant c$.*
(ii) *If N is a normal ϕ subgroup of G, G/N is ϕ nilpotent of finite class $\leqslant c$.*

Proof. (i) We prove by induction that $Z_j(G) \cap S \subseteq Z_j(S)$ ($j = 0, 1, \ldots, c$). Since $Z_\phi(G) \cap S \subseteq Z_\phi(S)$, the assertion is true for $j = 0$. We assume that $Z_i(G) \cap S \subseteq Z_i(S)$ and show that

$$Z_{i+1}(G) \cap S \subseteq Z_{i+1}(S).$$

Let z and s be elements of $Z_{i+1}(G) \cap S$ and S respectively; then $-s-z+s+z$ is in S, and is in $Z_i(G)$, since

$$(G, Z_{i+1}(G)) \subseteq Z_i(G).$$

Hence $-s-z+s+z$ is an element of $Z_i(G) \cap S \subseteq Z_i(S)$ so that z is in $Z_{i+1}(S)$ and $Z_{i+1}(G) \cap S \subseteq Z_{i+1}(S)$.

(ii) Since G is ϕ nilpotent of finite class c, $C^c(G) = 0$. By Lemma 4.2,

$$C^c(G/N) = \{C^c(G), N\}/N = N/N.$$

Hence G/N is ϕ nilpotent of finite class $\leqslant c$.

THEOREM 4.6. *Let G be an M-ϕ group. G is ϕ nilpotent of finite class if and only if a central chain connecting 0 and G may be obtained from any normal ψ chain for G by a suitable refinement.*

Proof. Assume that $Z_c(G) = G$ and let

$$0 = N_0 \subseteq \ldots \subseteq N_i \subseteq N_{i+1} \subseteq \ldots \subseteq N_n = G \tag{14}$$

be any ψ chain for G. Consider the chain

$$\begin{aligned} 0 \subseteq \ldots \subseteq N_i = \{N_i, Z_0 \cap N_{i+1}\} &\subseteq \ldots \subseteq \{N_i, Z_j \cap N_{i+1}\} \\ \subseteq \{N_i, Z_{j+1} \cap N_{i+1}\} &\subseteq \ldots \subseteq N_{i+1} \\ = \{N_i, Z_c \cap N_{i+1}\} = \{N_{i+1}, Z_0 \cap N_{i+2}\} &\subseteq \ldots \subseteq G. \end{aligned} \tag{15}$$

Clearly $\{N_i, Z_j \cap N_{i+1}\}$ is normal in G. Furthermore,

$$\{N_i, Z_{j+1} \cap N_{i+1}\}/\{N_i, Z_j \cap N_{i+1}\} \subseteq Z_\phi(G/\{N_i, Z_j \cap N_{i+1}\}),$$

as can be seen by using properties (k_4) and (k_6). Hence (15) is a central chain. This proves that the condition is necessary. The sufficiency is obvious.

COROLLARY 4.2. *Assume that the M-ϕ group G is ϕ nilpotent of finite class. If the ψ subgroups of G satisfy the double chain condition, then a ψ composition series is necessarily a central chain.*

5. M-ϕ groups with a finite Loewy chain. We now consider an M-ϕ group G which has a finite Loewy chain connecting 0 and G and show that in this case the upper and lower central chains are finite. Furthermore, if G is ϕ nilpotent of finite class, the upper Loewy chain, if it exists, is a central chain.

Definition. Let G be an M-ϕ group. If τ is the first ordinal such that $Z_\tau(G) = Z_{\tau+1}(G)$, then $Z_\tau(G)$ is the *hypercentre* of G and is denoted by $H(G)$. If σ is the first ordinal such that $C^\sigma(G) = C^{\sigma+1}(G)$, then $C^\sigma(G)$ is the *hypercommutator* of G and is denoted by $H^*(G)$. G is *ϕ nilpotent* if $H(G) = G$ and $H^*(G) = 0$.

Let us suppose for the moment that G is an M-ϕ group whose ψ subgroups satisfy the double chain condition. Then the hypercentre $H(G) = Z_n(G)$ for some integer n, and the hypercommutator $H^*(G) = C^m(G)$ for some integer m. Hence G is ϕ nilpotent if and only if G is ϕ nilpotent of finite class so that either of the following conditions is necessary and sufficient for G to be ϕ nilpotent:

$$\text{(i)}\ H(G) = G \quad \text{or} \quad \text{(ii)}\ H^*(G) = 0.$$

In this section we shall show that these results hold for an M-ϕ group which possesses a Loewy chain connecting 0 and G. Furthermore, if G is ϕ nilpotent then any Loewy chain connecting 0 and G (if one exists) is a central chain. This is an analogue to Corollary 4.2, which asserts that a ψ composition series (if one exists) is a central chain.

THEOREM 5.1. *Let J be a minimal normal ϕ subgroup of the M-ϕ group G which is not contained in the hypercommutator of G, then J is contained in the ϕ centre of G.*

Proof. J is contained in $G = C^0(G)$ but is not contained in $H^*(G) = C^\sigma(G)$. Hence there exists a first ordinal ν such that J is not contained in $C^\nu(G)$. Since $J \subseteq C^\mu(G)$ for all $\mu < \nu$ implies

$$J \subseteq \bigcap_{\mu<\nu} C^\mu(G),$$

ν is not a limit ordinal. Let $\nu = \lambda + 1$. Thus J is contained in $C^\lambda(G)$ but not in $C^{\lambda+1}(G)$.

Since $C^{\lambda+1}(G)$ is normal in G, $J \cap C^{\lambda+1}(G)$ is a normal ϕ subgroup of G. $J \cap C^{\lambda+1}(G)$ is contained in the minimal normal ϕ subgroup J and is not equal to J, since J is not contained in $C^{\lambda+1}(G)$. Hence

$$J \cap C^{\lambda+1}(G) = 0.$$

Let g be an element of G, and a an element of J; then

$$-g-a+g+a = (-g-a+g)+a$$

is in J, and is also in $C^{\lambda+1}(G)$, since $J \subseteq C^{\lambda}(G)$. Therefore $-g-a+g+a=0$, or a commutes with g. Thus $J \subseteq Z_{\phi}(G)$.

COROLLARY 5.1. *If $S(G)$ is the socle of the M-ϕ group G,*

$$S(G) \subseteq Z_{\phi}(G) + H^{*}(G).$$

In particular, if G is ϕ nilpotent, $S(G) \subseteq Z_{\phi}(G)$.

COROLLARY 5.2. *If, for every ψ subgroup N of the M-ϕ group G, G/N is ϕ nilpotent, any Loewy chain is a central chain. In particular, if G is ϕ nilpotent of finite class, every Loewy chain is a central chain.*

Proof. Let

$$(16) \qquad 0 = N_0 \subseteq \ldots \subseteq N_i \subseteq N_{i+1} \subseteq \ldots$$

be a Loewy chain. Then $N_{i+1}/N_i \subseteq S(G/N_i) \subseteq Z_{\phi}(G/N_i)$, since G/N_i is ϕ nilpotent. Hence (16) is a central chain.

THEOREM 5.2. *If the M-ϕ group G has a Loewy chain of length n which connects 0 and G, then $H^{*}(G) = C^{n}(G)$.*

Proof. From the theory of Loewy chains we know that the upper Loewy chain connects 0 and G and has length $\leq n$. Let

$$0 = S_0 \subset \ldots \subset S_j \subset \ldots \subset G$$

be the upper Loewy chain. For each positive integer i, $G/C^{i}(G)$ is ϕ nilpotent of finite class since, by Lemma 4.2,

$$C^{i}(G/C^{i}(G)) = \{C^{i}(G), C^{i}(G)\}/C^{i}(G) = C^{i}(G)/C^{i}(G).$$

The chain

$$\{S_0, C^{i}(G)\}/C^{i}(G) \subseteq \ldots \subseteq \{S_j, C^{i}(G)\}/C^{i}(G) \subseteq \{S_{j+1}, C^{i}(G)\}/C^{i}(G) \subseteq \ldots$$
$$\subseteq \{S_n, C^{i}(G)\}/C^{i}(G) = G/C^{i}(G)$$

is a Loewy chain (of length $\leq n$), since

$$\frac{\{S_{j+1}, C^{i}(G)\}/C^{i}(G)}{\{S_j, C^{i}(G)\}/C^{i}(G)} \simeq \frac{\{S_{j+1}, C^{i}(G)\}}{\{S_j, C^{i}(G)\}} \qquad (M\text{-}\phi),$$

which is contained in the socle of $G/\{S_j, C^{i}(G)\}$. By Corollary 5.2, this is a central chain for $G/C^{i}(G)$. Hence $G/C^{i}(G)$ is ϕ nilpotent of finite class $\leq n$. Therefore

$$C^{n}(G/C^{i}(G)) = C^{i}(G)/C^{i}(G).$$

But on the other hand, by Lemma 4.2, $C^{n}(G/C^{i}(G)) = \{C^{n}(G), C^{i}(G)\}/C^{i}(G)$. Thus $C^{n}(G) = C^{i}(G)$, for $i \geq n$, and $H^{*}(G) = C^{n}(G)$.

COROLLARY 5.3. *If the M-ϕ group G has a Loewy chain of length n which connects 0 and G, and if $H^{*}(G) = 0$, then G is ϕ nilpotent of finite class $\leq n$.*

A theorem about maximal normal ϕ subgroups analogous to Theorem 5.1 about minimal normal ϕ subgroups is:

THEOREM 5.3. *If J is a maximal normal ϕ subgroup of the M-ϕ group G which does not contain the hypercentre of G, then J contains $C^1(G)$.*

Proof. $H(G) = Z_\tau(G)$ is not contained in J. Hence there exists a first ordinal ν such that $Z_\nu(G)$ not $\subseteq J$. Since $Z_\mu(G) \subseteq J$ for all $\mu < \nu$,

$$\bigcup_{\mu<\nu} Z_\mu(G) \subseteq J$$

and therefore ν is not a limit ordinal. Let $\nu = \kappa + 1$. $Z_\kappa(G)$ is contained in J but $Z_{\kappa+1}(G)$ is not contained in J. $Z_{\kappa+1}(G)$ is normal in G, and J is a maximal normal ϕ subgroup of G; therefore

$$G = J + Z_{\kappa+1}(G).$$

Let z and z' be elements of $Z_{\kappa+1}(G)$; the element $-z-z'+z+z'$ is in $Z_\kappa(G) \subseteq J$. Hence G/J is abelian, or $C^1(G) \subseteq J$.

In Theorem 5.2 we have given a sufficient condition that the hypercommutator, $H^*(G)$ equal $C^n(G)$ for some integer n. We now find that under a somewhat weaker condition the hypercentre, $H(G)$ equals $Z_m(G)$ for some integer m. We need first a lemma.

LEMMA 5.1. *If the normal ϕ subgroup N of the M-ϕ group G is contained in $Z_r(G)$ for some integer r, and if G possesses a chain*

$$G = D_0 \supset \ldots \supset D_i \supset D_{i+1} \supset \ldots \supset D_m = 0,$$

where D_{i+1} is the intersection of maximal ψ subgroups of D_i, then

$$N \cap D_i / N \cap D_{i+1} \subseteq Z_\phi(G/N \cap D_{i+1}).$$

Proof. For fixed i $(0 < i < m)$ consider the chain

$$0 = Z_0(G) \cap N \cap D_i \subseteq \ldots \subseteq Z_j(G) \cap N \cap D_i \subseteq \ldots$$
$$\subseteq Z_r(G) \cap N \cap D_i = N \cap D_i.$$

If J is a maximal ψ subgroup of $N \cap D_i$, J contains the first subgroup of the chain but does not contain the last. Hence there exists an integer j such that

$$Z_j(G) \cap N \cap D_i \subseteq J; \; Z_{j+1}(G) \cap N \cap D_i \text{ not } \subseteq J.$$

Thus $J \subset (Z_{j+1}(G) \cap N \cap D_i) + J \subseteq N \cap D_i$, and, since J is maximal,

$$N \cap D_i = (Z_{j+1}(G) \cap N \cap D_i).$$

Let g and z be elements of G and $Z_{j+1}(G) \cap N \cap D_i$ respectively. The element $-g-z+g+z$ is in $N \cap D_i$ (since N and D_i are normal subgroups of G), and is also in $Z_j(G)$, since by definition

$$Z_{j+1}(G)/Z_j(G) = Z_\phi(G/Z_j(G)).$$

Thus $-g-z+g+z$ is in $Z_j \cap N \cap D_i \subseteq J$. Hence $J \supseteq (G, N \cap D_i)$. But $N \cap D_{i+1}$ is the intersection of maximal ψ subgroups of $N \cap D_i$. Therefore

$$N \cap D_{i+1} \supseteq (G, N \cap D_i)$$

and thus by Lemma 4.1, $N \cap D_i/N \cap D_{i+1} \subseteq Z_\phi(G/N \cap D_{i+1})$.

THEOREM 5.4. *If the M-ϕ group G possesses a chain*

$$G = D_0 \supset \ldots \supset D_i \supset D_{i+1} \supset \ldots \supset D_m = 0,$$

where D_{i+1} is the intersection of maximal ψ subgroups of D_i, then $H(G) = Z_m(G)$.

Proof. Let r be any positive integer and let $Z_r = Z_r(G)$. Then by Lemma 5.1,

$$Z_r \cap D_i/Z_r \cap D_{i+1} \subseteq Z_\phi(G/Z_r \cap D_{i+1}).$$

Hence

$$0 = N_0 = Z_r \cap D_m \subseteq \ldots \subseteq N_j = Z_r \cap D_{m-j} \subseteq \ldots \subseteq N_m = Z_r \cap D_0 = Z_r$$

is a central chain for G; and by Theorem 4.2, $Z_r = N_m \subseteq Z_m$. But r was arbitrary so that the relation holds for each r. Hence $Z_m = Z_r$ for $r > m$, and $H(G) = Z_m(G)$.

COROLLARY 5.4. *If the M-ϕ group G possesses a Loewy chain of length n which connects* 0 *and G, $H(G) = Z_n(G)$. Hence if $H(G) = G$, G is ϕ nilpotent of finite class $< n$, and the Loewy chain is a central chain.*

Proof. By Theorem 3.7, if G has a Loewy chain of length n connecting 0 and G, and if we define the chain

$$G = M_0 \supseteq \ldots \supseteq M_j \supseteq M_{j+1} \supseteq \ldots,$$

where M_{j+1} is the intersection of M_j with all maximal ψ subgroups of M_j, there exists an integer $m < n$ such that $M_m = 0$. Thus by Theorem 5.4, $H(G) = Z_m(G)$. But $n > m$, so that $H(G) = Z_n(G)$.

6. **ϕ-solubility.** In this section we study another property of the type discussed in §3. However, before defining the property, we prove some further results about ϕ nilpotency which we shall need.

LEMMA 6.1. *Let G be an M-ϕ group and assume that ϕ is normal. If N is a normal ϕ subgroup of G which is ϕ nilpotent of finite class, N is ψ nilpotent of finite class.*

Proof. It is sufficient to show that the ϕ subgroups $Z_i(N)$ are normal in G. To show that $Z_\phi(N)$ is normal in G, we note that $Z_\phi(N)$ is a subgroup of the centre $Z(N)$ of N, and that $Z(N)$ as a characteristic subgroup of N is normal in G. Hence if g is any element of G,

$$-g + Z_\phi(N) + g \subseteq -g + Z(N) + g = Z(N).$$

Since ϕ is normal, $-g + Z_\phi(N) + g$ is a ϕ subgroup of G; hence

$$-g + Z_\phi(N) + g = Z_\phi(N).$$

It may be shown by induction that $Z_i(N)$ is normal in G.

THEOREM 6.1. *Let G be an M-ϕ group and assume that ϕ is normal. If M and*

N are normal ϕ subgroups of G which are ϕ nilpotent of finite class, then $M + N$ is ϕ nilpotent of finite class.

Proof. (i) Assume that $M \cap N = 0$ so that $M + N = M \oplus N$. Since (by Lemma 6.1) M and N are ψ nilpotent, there exist chains:

$$0 = N_0 \subseteq \ldots \subseteq N_i \subseteq N_{i+1} \subseteq \ldots \subseteq N_n = N, \tag{17}$$

$$0 = M_0 \subseteq \ldots \subseteq M_i \subseteq M_{i+1} \subseteq \ldots \subseteq M_n = M, \tag{18}$$

with N_i and M_i ψ subgroups of G ($i = 1, \ldots, n$), and

$$N_{i+1}/N_i \subseteq Z_\phi(N/N_i); \quad M_{i+1}/M_i \subseteq Z_\phi(M/M_i).$$

Let m_{i+1}, n_{i+1}, m, and n be elements of M_{i+1}, N_{i+1}, M, and N respectively; the element

$$-(m+n) - (m_{i+1} + n_{i+1}) + (m+n) + (m_{i+1} + n_{i+1})$$

$$= -m - m_{i+1} + m + m_{i+1} - n - n_{i+1} + n + n_{i+1}$$

is in $M_i + N_i$, since $(M, M_{i+1}) \subseteq M_i$ and $(N, N_{i+1}) \subseteq N_i$. Hence

$$M_{i+1} + N_{i+1}/M_i + N_i \subseteq Z_\phi(M + N/M_i + N_i),$$

and the chain $0 = M_0 + N_0 \subseteq \ldots \subseteq M_i + N_i \subseteq M_{i+1} + N_{i+1} \subseteq \ldots \subseteq M + N$ is a central chain for $M + N$; thus $M + N$ is ϕ nilpotent of finite class.

(ii) We consider the general case (i.e., no longer assume that $M \cap N = 0$). Since $M/M \cap N$ and $N/M \cap N$ are ϕ nilpotent of finite class, it follows from (i) that $M + N/M \cap N$ is ϕ nilpotent of finite class and hence there exists a chain

$$M \cap N = Q_0 \subseteq \ldots \subseteq Q_i \subseteq Q_{i+1} \subseteq \ldots \subseteq Q_q = M + N, \tag{19}$$

where Q_i is in ψ, and $Q_{i+1}/Q_i \subseteq Z_\phi(M + N/Q_i)$. By Theorem 4.6, there exists a chain

$$0 = K_0 \subseteq \ldots \subseteq K_j \subseteq K_{j+1} \subseteq \ldots \subseteq K_k = M \cap N, \tag{20}$$

where K_j is in ψ and $K_{j+1}/K_j \subseteq Z_\phi(M/K_j)$, and there exists a refinement of (20):

$$0 = K_0 = K_{0,0} \subseteq \ldots \subseteq K_j = K_{j,0} \subseteq \ldots \subseteq K_{j,p} \subseteq \ldots \subseteq K_{j,n_j} \tag{21}$$

$$= K_{j+1} \subseteq \ldots \subseteq M \cap N,$$

where $K_{j,p}$ is in ψ, and $K_{j,p+1}/K_{j,p} \subseteq Z_\phi(N/K_{j,p})$. Clearly,

$$K_{j,p+1}/K_{j,p} \subseteq Z_\phi(M + N/K_{j,p}).$$

Combining (19) and (21) we obtain the chain

$$0 = K_0 \subseteq \ldots \subseteq K_{j,p} \subseteq \ldots \subseteq M \cap N = Q_0 \subseteq \ldots \subseteq Q_i \subseteq \ldots \subseteq Q_q$$

$$= M + N.$$

This is a central chain for $M + N$. Thus $M + N$ is ϕ nilpotent of finite class.

COROLLARY 6.1. *Let G be an M-ϕ group and assume that ϕ is normal. If the*

ψ subgroups of G satisfy the ascending chain condition, the compositum of normal ϕ nilpotent ϕ subgroups of finite class is ϕ nilpotent of finite class.

This result can also be obtained under the hypothesis that there exists a Loewy chain connecting 0 and G.

THEOREM 6.2. *Assume that the M-ϕ group G possesses a Loewy chain connecting 0 and G, and assume that ϕ is normal. If A_α, for each α in a set $\mathfrak{A}$, is a normal ϕ subgroup of G which is ϕ nilpotent of finite class, then $\complement A_\alpha (\alpha \in \mathfrak{A})$ is ϕ nilpotent of finite class.*

Proof. Let $0 = S_0 \subseteq \ldots \subseteq S_i \subseteq \ldots \subseteq S_n = G$ be a Loewy chain for G. If

$$A = \complement A_\alpha \qquad (\alpha \in \mathfrak{A})$$

the chain

$$(22) \quad 0 = A \cap S_0 = T_0 \subseteq \ldots \subseteq A \cap S_i = T_i \subseteq \ldots \subseteq A \cap S_n = T_n = A$$

is a Loewy chain for the M-ψ group A, that is, each T_{i+1}/T_i is the sum of minimal ψ subgroups of A/T_i. For

$$\frac{T_{i+1}}{T_i} = \frac{A \cap S_{i+1}}{A \cap S_i} \cong \frac{\{A \cap S_{i+1}, S_i\}}{S_i} \qquad (M\text{-}\psi),$$

which is the sum of minimal ψ subgroups since it is contained in S_{i+1}/S_i. We now show that the chain (22) is a central chain.

By Lemma 3.1, T_{i+1}/T_i can be written as the direct sum of minimal ψ subgroups; let

$$T_{i+1}/T_i = \sum{}^{\circ} F_\lambda/T_i \qquad (\gamma \in \mathfrak{C}),$$

where F_γ/T_i, for each γ in the set $\mathfrak{C}$, is a minimal ψ subgroup. For fixed γ in $\mathfrak{C}$ and for fixed α in $\mathfrak{A}$, we show that $T_i \supseteq (F_\gamma, A_\alpha)$. Since F_γ/T_i is a minimal ψ subgroup, either

$$F_\gamma \cap (A + T_i) = T_i \quad \text{or} \quad F_\gamma \cap (A + T_i) = F_\gamma.$$

In the first case, $F_\gamma \cap A_\alpha \subseteq T_i$; and $(F_\gamma, A_\alpha) \subseteq F_\gamma \cap A_\alpha$ so that $(F_\gamma, A_\alpha) \subseteq T_i$. In the second case,

$$F_\gamma \subseteq A_\alpha + T_i \quad \text{or} \quad F_\gamma/T_i \subseteq A_\alpha + T_i/T_i.$$

Since F_γ/T_i is a minimal ψ subgroup of the ψ nilpotent group $A_\alpha + T_i/T_i$, it is contained in $Z_\phi(A_\alpha + T_i/T_i)$. Therefore, $(A_\alpha + T_i, F_\gamma) \subseteq T_i$ so that $(A_\alpha, F_\gamma) \subseteq T_i$. It follows that, for each γ in $\mathfrak{C}$ and for each α in $\mathfrak{A}$, $(A_\alpha, F_\gamma) \subseteq T_i$. It follows that, for each γ in $\mathfrak{C}$, $(\complement A_\alpha, F_\gamma) \subseteq T_i$ or equivalently, $F_\gamma/T_i \subseteq Z_\phi(A/T_i)$. This in turn implies that

$$T_{i+1}/T_i = \sum{}^{\circ} F_\gamma/T_i \subseteq Z_\phi(A/T_i),$$

which shows that (22) is a central chain. Hence $A = \complement A_\alpha$ is ϕ nilpotent of finite class.

THEOREM 6.3. *Assume that the hypercommutator $H^*(G)$ of the M-ϕ group G, is equal to $C^n(G)$ for some integer n. If N_a is a normal ϕ subgroup of G, for each a in the set $\mathfrak{A}$, and if G/N_a is ϕ nilpotent of finite class, then $G/\bigcap N_a$ $(a \in \mathfrak{A})$ is ϕ nilpotent of finite class.*

Proof. There exists a central chain for G of finite length n_a connecting N_a to G, for each a. Hence

$$C^{n_a}(G) \subseteq N_a.$$

But

$$H^*(G) = C^n(G) \subseteq C^{n_a}(G)$$

for each a. Hence $H^*(G) \subseteq N_a$, for each a, and

$$H^*(G) \subseteq \bigcap_a N_a = N.$$

$G/H^*(G)$ is ϕ nilpotent of finite class and hence G/N is ϕ nilpotent of finite class.

COROLLARY 6.2. *Under the hypotheses of the previous theorem, $H^*(G)$ is the intersection of all normal ϕ subgroups N such that G/N is ϕ nilpotent of finite class.*

LEMMA 6.2. *Let A and B be normal ϕ subgroups of the M-ϕ group G with $A \supset B$. If A/B is ϕ nilpotent of finite class, $A\eta/B\eta$ is ϕ nilpotent of finite class for any M-ϕ homomorphism η of G.*

Proof. There exists a chain $B = B_0 \subseteq \ldots \subseteq B_i \subseteq B_{i+1} \subseteq \ldots \subseteq B_n = A$, where B_i is a normal ϕ subgroup of A and $B_{i+1}/B_i \subseteq Z_\phi(A/B_i)$. By Theorem 4.1,

$$B_{i+1}\eta/B_i\eta \subseteq Z_\phi(A\eta/B_i\eta),$$

and thus $A\eta/B\eta$ is ϕ nilpotent of finite class.

It may be shown in a similar fashion that the following is a consequence of Theorem 4.3 (ii):

LEMMA 6.3. *Let A, B, and C be normal ϕ subgroups of the M-ϕ group G with $A \supset B$. If A/B is ϕ nilpotent of finite class, then $A \cap C/B \cap C$ is ϕ nilpotent of finite class.*

Consider now the property (S) of M-ϕ groups:

(S) The ϕ subgroup A of the M-ϕ group G satisfies (S) (in G), if it is ϕ nilpotent of finite class.

In order to apply our theory of normal chains we must verify that (S) satisfies the conditions (k_1)-(k_6). (k_1) obviously holds. The validity of (k_4) follows from Lemma 6.3. Lemma 6.2 shows that (k'_5) holds; and (k'_5) is equivalent to (k_5) and (k_6). In order to ensure that (k_2) and (k_3) hold we make further hypotheses about the groups under consideration.

Assume that ϕ is normal. It follows from Corollary 6.1 that (k_2) is satisfied if the ascending chain condition holds for the ψ subgroups. On the other hand, in virtue of Theorem 6.3, (k_3) is satisfied if the descending chain condition holds

for the ψ subgroups. Hence (k_2) and (k_3) hold if we assume the double chain condition for ψ subgroups. However, this condition may be replaced by the weaker condition that G possesses a Loewy chain connecting 0 and G. This follows from Theorem 6.2 (for (k_2)); and from Theorems 5.2 and 6.3 (for (k_3)). So we have:

THEOREM 6.4. *Let G be an M-ϕ group. Assume that ϕ is normal and that G possesses a Loewy chain connecting* 0 *and G. Then* (S) *satisfies* (k_1)-(k_6).

Therefore, the upper and lower S-chains may be constructed, and the results of §2 hold for S-chains. The terms of the lower S-chain are:

$$G \supseteq H^*(G) \supseteq H_2^*(G) = H^*[H^*(G)] \supseteq \dots \supseteq H_{n+1}^*(G) = H^*[H_n^*(G)] \supseteq \dots .$$

This follows from Corollary 6.2. However, the terms of the upper S-chain are not necessarily the successive hypercentres, for the hypercentre $H(G)$ is not necessarily the maximal ϕ nilpotent normal ϕ subgroup of G.

Definition. If the *M-ϕ group G* possesses an S-chain that connects 0 and G, G is *ϕ soluble.*

THEOREM 6.5. *Let G be an M-ϕ group. Assume that ϕ is normal and that G possesses a Loewy chain connecting* 0 *and G. If G is ϕ soluble, any Loewy chain connecting* 0 *and G has abelian factors and consequently is an* S*-chain.*

Proof. Let $0 = U_0 \subseteq \dots \subseteq U_i \subseteq U_{i+1} \subseteq \dots \subseteq U_m = G$ be a Loewy chain for G; then U_{i+1}/U_i is the direct sum of feet of G/U_i. Hence in order to show that U_{i+1}/U_i is abelian, it is sufficient to show that any foot of G/U_i is abelian.

Let F/U_i be a foot of G/U_i. Since G is ϕ soluble, there exists a chain

$$U_i = T_0 \subseteq \dots \subseteq T_j \subseteq T_{j+1} \subseteq \dots \subseteq T_m = G,$$

where T_j is in ψ and T_{j+1}/T_j is ψ nilpotent of finite class. Choose j so that F is not contained in T_j but is contained in T_{j+1}. Then

$$U_i \subseteq F \cap T_j \subset F$$

and hence, since F/U_i is a minimal ψ subgroup, $U_i = F \cap T_j$. Now $F + T_j/T_j$ is a minimal ψ subgroup of the ψ nilpotent group T_{j+1}/T_j; by Corollary 5.1, $F + T_j/T_j$ is in the centre of T_{j+1}/T_j. This implies that F/U_i is abelian, since

$$F/U_i \cong F + T_j/T_j.$$

The definition of solubility that we have used was discussed by Hirch [6]. It is customary to proceed somewhat differently.

Definition. For the M-ϕ group G we define

$$G^{(0)} = G, G^{(n+1)} = (G^{(n)}, G^{(n)}),$$

for $n > 0$.

$$G = G^{(0)} \supseteq \dots \supseteq G^{(i)} \supseteq G^{(i+1)} \supseteq \dots \tag{23}$$

is a descending normal ϕ chain, and the factors $G^{(i+1)}/G^{(i)}$ are abelian; in fact,

$G^{(i+1)}$ is the smallest normal ϕ subgroup of $G^{(i)}$ such that the quotient group is abelian. However the dual construction does not yield an ascending normal ϕ chain with abelian factors; for the compositum of abelian normal ϕ subgroups is not necessarily abelian.

The following theorem shows that the definition given for ϕ solubility coincides with the customary one:

THEOREM 6.6. *The M-ϕ group G is ϕ soluble, if and only if $G^{(s)} = 0$ for some integer s.*

Proof. The chain $G = G^{(0)} \supset \ldots \supset G^{(i)} \supset \ldots \supset G^{(s)} = 0$ has abelian factors and hence is an S-chain.

Conversely, assume that $G = R_0 \supseteq \ldots \supseteq R_i \supseteq R_{i+1} \supseteq \ldots \supseteq R_n = 0$ is an S-chain so that R_i/R_{i+1} is ϕ nilpotent of finite class. Then the chain

$$R_i/R_{i+1} = C^0(R_i/R_{i+1}) \supseteq \ldots \supseteq C^j(R_i/R_{i+1}) \supseteq \ldots \supseteq C^{n_i}(R_i/R_{i+1})$$
$$= R_{i+1}/R_{i+1}$$

joins R_i/R_{i+1} to R_{i+1}/R_{i+1} and has abelian factors. Hence if

$$C^j(R_i/R_{i+1}) = R_{i,j}/R_{i+1}, \quad R_{i,n_i} = R_{i+1},$$

the chain

$$G = R_0 \supseteq \ldots \supseteq R_i \supseteq \ldots \supseteq R_{i,j} \supseteq \ldots \supseteq R_{i,n_i} = R_{i+1} \supseteq \ldots \supseteq R_n = 0$$

is a normal ϕ chain for G with abelian factors. It is easy to verify that if there exist a normal ϕ chain with abelian factors connecting G and 0, then $G^{(s)} = 0$ for some integer s.

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QUELQUES REMARQUES SUR LA GÉNÉRALISATION DU SCALAIRE DE COURBURE ET DU SCALAIRE PRINCIPAL

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Introduction. Dans un espace finslérien à n dimensions rapporté à un système quelconque de coordonnées $x^1, x^2, \ldots, x^n$, la distance de deux points infiniment voisins est donnée par la formule

$$ds = F(x^1, x^2, \ldots, x^n, dx^1, dx^2, \ldots, dx^n) \tag{1}$$

où la fonction $F(x^2, x^2, \ldots, x^n, dx^1, \ldots, dx^n)$ est positivement homogène et du premier degré par rapport aux dx^i. La longueur d'un arc d'une courbe

$$x^i = x^i(t) \qquad (i = 1, 2, \ldots, n) \tag{2}$$

est définie entre $x^i(t_0)$ et $x^i(t_1)$, d'après (1), par

$$s = \int_{t_0}^{t_1} F(x, x')dt. \tag{3}$$

L'expression (3) détermine une géométrie dont l'élément fondamental est l'élément d'appui (x, x'). Les tenseurs et les invariants caractéristiques de l'espace finslérien dépendent alors de (x, x'). L'espace est un ensemble des éléments d'appui (x, x') [3].

Dans ses mémoires [1] et [2], Berwald a donné deux scalaires caractéristiques de l'espace finslérien à deux dimensions: la scalaire de courbure $\mathfrak{R}$ (il a désigné le scalaire de courbure par $\mathfrak{K}$) et le scalaire principal $\mathfrak{J}$. Les expressions analytiques de $\mathfrak{R}$ et de $\mathfrak{J}$ sont:

$$\mathfrak{R} = \tfrac{1}{2} R_0{}^i{}_{jk} h_i (l^j h^k - l^k h^j), \tag{4a}$$

$$\mathfrak{J} = \tfrac{1}{2} A_{ijk} h^i h^j h^k, \tag{4b}$$

où $R_0{}^i{}_{jk}$ est le tenseur de courbure riemannien contracté par le vecteur l^s $(R_0{}^i{}_{jk} = R_s{}^i{}_{jk}\, l^s)$, A_{ijk} est le tenseur de torsion de l'espace, l^i est le vecteur unitaire porté dans la direction de son élément d'appui et h^i est le vecteur normal unitaire.

Dans mon article [5], j'ai exprimé le vecteur normal h^i par le vecteur d'Euler:

$$\rho_i = \frac{\partial F}{\partial x^i} - \frac{d}{dt} F_{x'^i} \tag{5}$$

sous la forme

$$h^i = -\frac{r}{F} \rho^i \tag{6}$$

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où

$$\frac{1}{r} = \mathfrak{L}(\rho^i) = \sqrt{g_{ik}\rho^i\rho^k} = \sqrt{\rho^i\rho_i}. \tag{6a}$$

$1/r$ est alors la longueur du vecteur ρ^i, et ainsi on peut déterminer les deux invariants $\mathfrak{R}$ et $\mathfrak{J}$ par les formules (4a), (4b) et (6) même dans l'espace à n dimensions, car le vecteur ρ^i est un vecteur de l'espace à n dimensions.

Dans la section 1 du présent article nous allons démontrer que $1/r$ est aussi un invariant géométrique à n dimensions (si $n = 2$, $1/r$ est la courbure extrémale) et que les scalaires

$$\mathfrak{R} = \frac{r^2}{2F^2} R_0{}^i{}_{jk}\rho_i(l^j\rho^k - l^k\rho^j)$$

et

$$\mathfrak{J} = -\frac{r^3}{2F^3} A_{ijk}\rho^i\rho^j\rho^k$$

dépendent de la courbe (2) si $n > 2$. Le vecteur unitaire[1]

$$\rho^{*i} = \mp r\rho^i$$

est le vecteur normal principal de la courbe (2).

Dans la section 2 nous donnerons la condition nécessaire et suffisante pour que les tenseurs $R_0{}^i{}_{jk}$ et A_{ijk} aient la forme tridimensionnelle:

$$R_0{}^i{}_{jk} = \mathfrak{R}\rho^{*i}(l_j\rho^*_k - l_k\rho^*_j), \tag{7a}$$

$$A_{ijk} = 2\mathfrak{J}\rho^*_i\rho^*_j\rho^*_k. \tag{7b}$$

Les domaines des espaces où les tenseurs $R_0{}^i{}_{jk}$ et A_{ijk} ont la forme donnée par (7a), (7b) sont de caractère semblable aux espaces à deux dimensions.

1. Interprétation géométrique du scalaire de courbure et du scalaire principal de l'espace à n dimensions. Considérons le scalaire de courbure $\mathfrak{R}$ et le scalaire principal $\mathfrak{J}$ d'un espace finslérien à n dimensions le long de la courbe

$$x^i = x^i(s) \qquad (i = 1, 2, \ldots, n). \tag{8}$$

Le paramètre s est comme d'ordinaire la longueur mesurée sur la courbe (8). On a alors:

$$F(x, x') = 1, \qquad x'^i = \frac{dx^i}{ds}. \tag{9}$$

D'après (9), le long de la courbe (8), les formules de $\mathfrak{R}$ et de $\mathfrak{J}$ sont les suivantes:

$$\mathfrak{R} = \tfrac{1}{2}r^2 R_0{}^i{}_{jk}\rho_i(l^j\rho^k - l^k\rho^j), \tag{10a}$$

$$\mathfrak{J} = -\tfrac{1}{2}r^3 A_{ijk}\rho^i\rho^j\rho^k, \tag{10b}$$

où ρ_i est le vecteur d'Euler (cf. l'équation (5)), et le scalaire $1/r$ est la longueur du vecteur ρ_i.

[1] ρ^{*i} est désigné dans [5] par σ^i.

Nous aurons, d'après (5), pour le vecteur ρ_i l'expression

$$\rho_i = \frac{\partial F}{\partial x^i} - \frac{\partial^2 F}{\partial x^j \partial x'^i} x'^j - \frac{\partial^2 F}{\partial x'^j \partial x'^i} x''^j. \tag{11}$$

On a, en vertu de la formule de Frenet, l'équation suivante:

$$Dx'^i = \frac{dx'^i}{ds} + C_j{}^i{}_k x'^j dx'^k + \Gamma_j{}^i{}_k x'^j dx^k = \kappa\eta^i, \tag{12}$$

où κ est la courbure de la courbe (8) et η^i est le vecteur normal principal de (8).

Les termes $C_j{}^i{}_k x'^j x''^k$ et $\Gamma_j{}^i{}_k x'^j x'^k$ ont la valeur [6]:

$$C_j{}^i{}_k x'^j x''^k = 0, \tag{13a}$$

$$\Gamma_j{}^i{}_k x'^j x'^k = 2G^i, \tag{13b}$$

où

$$G^i = g^{ri} G_r, \qquad G_r = \frac{1}{4}\left(\frac{\partial^2 F^2}{\partial x'^r \partial x^k} x'^k - \frac{\partial F^2}{\partial x^r}\right). \tag{14}$$

Nous aurons alors d'après (12), étant donné (13a) et (13b),

$$\frac{dx'^i}{ds} = x''^i = \kappa\eta^i - 2G^i. \tag{15}$$

A cause de la homogénéité de G^i on a encore [6]:

$$2G^i = \frac{\partial G^i}{\partial x'^j} x'^j.$$

La définition du tenseur métrique nous donne l'expression:

$$g_{ij} = F\frac{\partial^2 F}{\partial x'^i \partial x'^j} + \frac{\partial F}{\partial x'^i}\,\frac{\partial F}{\partial x'^j}$$

où, à l'aide de (9), le long de la courbe (8) nous aurons

$$\frac{\partial^2 F}{\partial x'^i \partial x'^j} = g_{ij} - l_i l_j \qquad \left(l_r = \frac{\partial F}{\partial x'^r}\right). \tag{16}$$

Nous avons encore la formule:

$$\frac{1}{2F}\,\frac{\partial^2 F^2}{\partial x'^i \partial x^j} x'^j = \frac{\partial^2 F}{\partial x^j \partial x'^i} x'^j + \frac{1}{F}\,\frac{\partial F}{\partial x^j}\,\frac{\partial F}{\partial x'^i} x'^j$$

et le long de (8) de nouveau en vertu de (9)

$$\frac{\partial^2 F}{\partial x^j \partial x'^i} x'^j = \frac{1}{2}\,\frac{\partial^2 F^2}{\partial x^j \partial x'^i} x'^j - \frac{\partial F}{\partial x^j} l_i l^j, \tag{17}$$

$$\frac{\partial F}{\partial x'^i} = l_i, \qquad x'^j = l^j. \tag{17a}$$

Nous pouvons maintenant exprimer ρ_i de l'équation (11) d'après (15), (16) et (17) sous la forme:

(18) $$\rho_i = \frac{1}{2}\left(\frac{\partial^2 F^2}{\partial x^i} - \frac{\partial^2 F^2}{\partial x^j \partial x'^i} x'^j\right) + \frac{\partial F}{\partial x^j} l_i l^j - (g_{ij} - l_i l_j)(\kappa\eta^j - 2G^j).$$

Le vecteur l_j est le vecteur tangent à la courbe (8).

$$l_j = \frac{\partial F}{\partial x'^j} \qquad (j = 1, 2, \ldots, n)$$

sont les composantes covariantes du vecteur tangent. Comme nous l'avons déjà remarqué, η^j est le vecteur normal principal de la courbe. Nous aurons par conséquent:

(19) $$l_j \eta^j = 0.$$

L'équation (18) nous donne le vecteur ρ_i d'après (14) et (19) sous la forme:

(20) $$\rho_i = \frac{\partial F}{\partial x^j} l_i l^j - \kappa\eta_i - 2G^j l_i l_j.$$

Calculons maintenant le terme $2G^j l_i l_j$. D'après (14) on a

$$2G^j l_i l_j = 2G_j l_i l^j = \frac{1}{2}\left(2\frac{\partial F^2}{\partial x^k} x'^k - \frac{\partial F^2}{\partial x^k} x'^k\right) l_i$$

et à l'aide de (9) et (17a),

(21) $$2G^j l_i l_j = \frac{\partial F}{\partial x^k} l_i l^k = \frac{\partial F}{\partial x^j} l_i l^j.$$

Les équations (20) et (21) nous donnent

(22) $$\rho_i = -\kappa\eta_i.$$

Le vecteur η_i étant un vecteur unitaire, la longueur du vecteur ρ_i sera d'après (22):

(23) $$\mathfrak{L}(\rho^i) = \sqrt{\rho^i \rho_i} = \kappa.$$

Le scalaire r dans les expressions (10a), (10b) est alors, d'après (6a) et (23), le rayon de courbure de la courbe (8),

(24) $$r = 1/\kappa.$$

La formule de Frenet:

$$\omega^i = Dx'^i = \kappa\eta^i$$

nous donne d'après les équations (22) et (24),

(25) $$\rho^i = -\omega^i = -\eta^i/r.$$

Donc ω^i est la différentielle absolue du vecteur x'^i.

On peut maintenant exprimer les scalaires $\mathfrak{R}$ et $\mathfrak{J}$ d'après (10a) et (10b) à l'aide de l'équation (25) par le vecteur normal unitaire ou par la différentielle absolue du vecteur $l^i = x'^i$ sous la forme

(26) $$\mathfrak{R} = \tfrac{1}{2}R_0{}^i{}_{jk}\eta_i(l^j\eta^k - l^k\eta^j) = \tfrac{1}{2}r^2 R_0{}^i{}_{jk}\omega_i(l^j\omega^k - l^k\omega^j),$$

(27) $$\mathfrak{J} = \tfrac{1}{2}A_{ijk}\eta^i\eta^j\eta^k = \tfrac{1}{2}r^3A_{ijk}\omega^i\omega^j\omega^k.$$

Dans l'espace à deux dimensions le vecteur η^i ne dépend que du vecteur $l^i = x'^i$; η^i est alors dans ce cas une fonction de (x, x'). Dans l'espace à n dimensions $(n > 2)$, η^i dépend non seulement de l'élément d'appui, mais aussi de la courbe

$$x^i = x^i(s) \qquad (i = 1, 2, \ldots, n).$$

C'est qu'une courbe à deux dimensions n'a qu'un vecteur normal, et ce vecteur dépend alors du vecteur l^i.

2. Les invariants $\mathfrak{R}$ et $\mathfrak{J}$ dans l'espace à trois dimensions. Dans cette partie, nous considérerons les tenseurs A_{ijk} et $R_0{}^i{}_{jk}$ dans l'espace finslérien à trois dimensions. D'abord nous allons construire les vecteurs ${}_{(\alpha)}\mu_i$ d'un trièdre. On aura alors [**4**]:

(28a) $${}_{(\alpha)}\mu_i\,{}_{(\alpha)}\mu^k = \delta_i^k$$

(28b) $${}_{(\alpha)}\mu_i\,{}_{(\beta)}\mu^i = \delta_{\alpha\beta} \qquad \delta_{rs} = \delta_r^s = \begin{cases}1, & r = s\\ 0, & r \neq s\end{cases}$$

${}_{(\alpha)}\mu^i$ $(i = 1, 2, 3)$ sont les composantes contravariantes et ${}_{(\alpha)}\mu_i$ les composantes covariantes des vecteurs ${}_{(\alpha)}\mu$.

On voit facilement que

(29) $$l^i\rho_i^* = 0,$$

où

(29a) $$\rho_i^* = r\rho_i, \qquad r = 1/\mathfrak{L}(\rho_i)$$

(cf. (6a)). Le vecteur ρ_i^* est un vecteur unitaire, parce qu'on a d'après (29a):

$$\mathfrak{L}(\rho^{*i}) = r\sqrt{g^{ik}\rho_i\rho_k} = r\mathfrak{L}(\rho_i) = 1.$$

Désignons par g le déterminant $|g_{ik}|$; les fonctions $\pm\sqrt{g}$ constituent un tenseur:

(30) $$\epsilon_{123} = \sqrt{g}; \quad \epsilon_{ijk} = -\epsilon_{jik}; \quad \epsilon_{ijk} = -\epsilon_{ikj}; \quad \epsilon_{rrs} = 0.$$

Toutes les composantes du tenseur ϵ sont définies par les formules (30). On voit facilement que ϵ_{ijk} est un tenseur. On peut toujours exprimer g_{ik} sous la forme:

(31) $$g_{ik} = {}_{(\alpha)}\mu_i\,{}_{(\alpha)}\mu_k.$$

On a alors d'après (31),

$$g = \begin{vmatrix} g_{11} & g_{12} & g_{13}\\ g_{21} & g_{22} & g_{23}\\ g_{31} & g_{32} & g_{33}\end{vmatrix} = \begin{vmatrix} {}_{(1)}\mu_1 & {}_{(1)}\mu_2 & {}_{(1)}\mu_3\\ {}_{(2)}\mu_1 & {}_{(2)}\mu_2 & {}_{(2)}\mu_3\\ {}_{(3)}\mu_1 & {}_{(3)}\mu_2 & {}_{(3)}\mu_3\end{vmatrix}^2.$$

$\pm\sqrt{g}$ est donc un tenseur de Plücker [**4**].

Les composantes contravariantes du tenseur ϵ sont les suivantes:

$$\epsilon^{rst} = g^{ri}g^{sj}g^{tk}\epsilon_{ijk} = \frac{1}{g^3}B_{ri}B_{sj}B_{tk}\epsilon_{ijk}$$

où B_{mn} est le déterminant complémentaire de g_{mn} dans $|g_{ik}|$. D'après l'identité

$$\epsilon_{ijk} = \pm\sqrt{g} = \begin{vmatrix} {}_{(1)}\mu_i & {}_{(1)}\mu_j & {}_{(1)}\mu_k \\ {}_{(2)}\mu_i & {}_{(2)}\mu_j & {}_{(2)}\mu_k \\ {}_{(3)}\mu_i & {}_{(3)}\mu_j & {}_{(3)}\mu_k \end{vmatrix}$$

nous aurons:

$$\epsilon^{rst} = \frac{1}{g^3}\begin{vmatrix} B_{ri}\,{}_{(1)}\mu_i & B_{sj}\,{}_{(1)}\mu_j & B_{tk}\,{}_{(1)}\mu_k \\ B_{ri}\,{}_{(2)}\mu_i & B_{sj}\,{}_{(2)}\mu_j & B_{tk}\,{}_{(2)}\mu_k \\ B_{ri}\,{}_{(3)}\mu_i & B_{sj}\,{}_{(3)}\mu_k & B_{tk}\,{}_{(3)}\mu_k \end{vmatrix} = \frac{1}{g^3}\begin{vmatrix} B_{r1} & B_{s1} & B_{t1} \\ B_{r2} & B_{s2} & B_{t2} \\ B_{r3} & B_{s3} & B_{t3} \end{vmatrix} \cdot \begin{vmatrix} {}_{(1)}\mu_1 & {}_{(1)}\mu_2 & {}_{(1)}\mu_3 \\ {}_{(2)}\mu_1 & {}_{(2)}\mu_2 & {}_{(2)}\mu_3 \\ {}_{(3)}\mu_1 & {}_{(3)}\mu_2 & {}_{(3)}\mu_3 \end{vmatrix}.$$

Si $r = 1$, $s = 2$, $t = 3$, nous aurons

$$\begin{vmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{vmatrix} = g^2$$

et ainsi

(32) $$\epsilon^{123} = 1/\sqrt{g},$$

(32a) $$\epsilon^{ijk} = -\epsilon^{jik}; \quad \epsilon^{ijk} = -\epsilon^{ikj}; \quad \epsilon^{rrs} = 0.$$

L'identité suivante

(33) $$\epsilon^{ijk}\epsilon_{irs} = \delta^j_r\delta^k_s - \delta^j_s\delta^k_r$$

nous permet de vérifier les équations:

(34) $$\sigma^{*i}\sigma^*_i = 1, \quad \sigma^{*i}l_i = \sigma^{*i}\rho^*_i = 0,$$
où

$$\sigma^{*i} = \epsilon^{ijk}l_j\rho^*_k, \quad \sigma^*_i = \epsilon_{ijk}l^j\sigma^{*k}.$$

Si l'on prend

(35) $${}_{(1)}\mu_i = l_i, \quad {}_{(2)}\mu_i = \rho^*_i, \quad {}_{(3)}\mu_i = \sigma^*_i,$$

les vecteurs l, ρ^* et σ^* satisfont les équations (28b), et en vertu d'un résultat connu du calcul des tenseurs [4, I §4], ils satisfont aussi l'équation (28a).

On peut donc réaliser le long d'une courbe, par les vecteurs l, ρ^*, σ^* et par quelques scalaires, tous les tenseurs de l'espace à trois dimensions. Nous aurons pour le tenseur $R_0{}^i{}_{jk}$ la formule:

(36) $$R_0{}^i{}_{jk} = \underset{(rst)}{\mathfrak{R}}\,{}_{(r)}\mu^i\,{}_{(s)}\mu_j\,{}_{(t)}\mu_k.$$

Les fonctions $\underset{(rst)}{\mathfrak{R}}$ sont des scalaires. D'après une contraction par ${}_{(a)}\mu_i$, ${}_{(b)}\mu^j$, ${}_{(c)}\mu^k$, il résulte de l'équation (36) à l'aide de (28b):

(36a) $$\underset{(abc)}{\mathfrak{R}} = R_0{}^i{}_{jk}\,{}_{(a)}\mu_i\,{}_{(b)}\mu^j\,{}_{(c)}\mu^k.$$

Pour le tenseur métrique g_{ik} on a le long de la courbe (8) la formule

$$g_{ik} = \underset{(rs)}{\mathfrak{G}}\,{}_{(r)}\mu_i\,{}_{(s)}\mu_k$$

et d'après une contraction basée sur (28b) nous aurons

$$\underset{(ab)}{\mathfrak{G}} = g_{ik}\,{}_{(a)}\mu^{i}\,{}_{(b)}\mu^{k} = \delta_{ab},$$

et alors

$$g_{ik} = l_i l_k + \rho^*_i\rho^*_k + \sigma^*_i\sigma^*_k.$$

On a l'identité

$$R_0{}^i{}_{jk} l_i = 0,$$

et nous aurons d'après (36a):

$$\underset{(1bc)}{\mathfrak{R}} = R_0{}^i{}_{jk} l_i\,{}_{(b)}\mu^{j}\,{}_{(c)}\mu^{k} = 0. \tag{37}$$

Le tenseur $R_0{}^i{}_{jk}$ est antisymétrique par rapport à ses deux derniers indices. Il en résulte:

$$\underset{(abc)}{\mathfrak{R}} = -R_0{}^i{}_{kj}\,{}_{(a)}\mu_i\,{}_{(b)}\mu^{j}\,{}_{(c)}\mu^{k} = -\underset{(acb)}{\mathfrak{R}}. \tag{37a}$$

D'après les équations (35), (37) et (37a), il résulte de (36):

$$\begin{aligned} R_0{}^i{}_{jk} = {} & \underset{(212)}{\mathfrak{R}}\rho^{*i}(l_j\rho^*_k - l_k\rho^*_j) + \underset{(213)}{\mathfrak{R}}\rho^{*i}(l_j\sigma^*_k - l_k\sigma^*_j) \\ & + \underset{(223)}{\mathfrak{R}}\rho^{*i}(\rho^*_j\sigma^*_k - \rho^*_k\sigma^*_j) + \underset{(312)}{\mathfrak{R}}\sigma^{*i}(l_j\rho^*_k - l_k\rho^*_j) \\ & + \underset{(313)}{\mathfrak{R}}\sigma^{*i}(l_j\sigma^*_k - l_k\sigma^*_j) + \underset{(323)}{\mathfrak{R}}\sigma^{*i}(\rho^*_j\sigma^*_k - \rho^*_k\sigma^*_j). \end{aligned} \tag{38}$$

Supposons maintenant que:

$$R_0{}^i{}_{jk}(l^j\sigma^{*k} - l^k\sigma^{*j}) = R_0{}^i{}_{jk}(\rho^{*j}\sigma^{*k} - \rho^{*k}\sigma^{*j}) = R_0{}^i{}_{jk}\sigma^*_i = 0. \tag{39}$$

On a alors d'après (36a) et (37a),

$$2\underset{(213)}{\mathfrak{R}}\rho^{*i} = 2\underset{(223)}{\mathfrak{R}}\rho^{*i} = \underset{(3ab)}{\mathfrak{R}}\,{}_{(a)}\mu_j\,{}_{(b)}\mu_k = 0 \qquad (a, b = 1, 2, 3).$$

et pour le tenseur $R_0{}^i{}_{jk}$:

$$R_0{}^i{}_{jk} = \underset{(212)}{\mathfrak{R}}\rho^{*i}(l_j\rho^*_k - l_k\rho^*_j). \tag{40}$$

Dans ce cas, le tenseur $R_0{}^i{}_{jk}$ aura la même forme que dans l'espace à deux dimensions.

Calculons maintenant le tenseur de torsion A_{ijk}. Le tenseur A_{ijk} aura la forme:

$$A_{ijk} = \underset{(rst)}{\mathfrak{J}}\,{}_{(r)}\mu_i\,{}_{(s)}\mu_j\,{}_{(t)}\mu_k. \tag{41}$$

Après une contraction par ${}_{(a)}\mu^i$, ${}_{(b)}\mu^j$ et ${}_{(c)}\mu^k$ il résulte de (41), étant donné (28b),

$$\underset{(abc)}{\mathfrak{J}} = A_{ijk}\,{}_{(a)}\mu^{i}\,{}_{(b)}\mu^{j}\,{}_{(c)}\mu^{k}. \tag{42}$$

A cause de l'homogénéité du premier degré par rapport aux x'^i de la fonction fondamentale $F(x, x')$ (cf. l'équation (3)) et d'après

$$A_{ijk} = \frac{F}{4}\,\frac{\partial^3 F^2}{\partial x'^i\,\partial x'^j\,\partial x'^k} = \frac{F}{2}\,\frac{\partial g_{ij}}{\partial x'^k}, \tag{43}$$

il résulte que

$$(44) \qquad A_{ijk}l^i = A_{ijk}l^j = A_{ijk}l^k = 0,$$

et que le tenseur A_{ijk} est symétrique par rapport à ses indices; c'est à dire:

$$(45) \qquad A_{ijk} = A_{jik} = A_{ikj} = \ldots .$$

Si l'on prend, comme précédemment pour ${}_{(1)}\mu$ le tenseur l, alors

$${}_{(1)}\mu_i = l_i, \qquad {}_{(1)}\mu^i = l^i,$$

et on aura pour les scalaires $\underset{(abc)}{\mathfrak{J}}$ d'après (44) et (42):

$$(46) \qquad \underset{(1bc)}{\mathfrak{J}} = \underset{(a1c)}{\mathfrak{J}} = \underset{(ab1)}{\mathfrak{J}} = 0.$$

Nous démontrerons encore que $\underset{(abc)}{\mathfrak{J}}$ est symétrique par rapport à a, b, c. C'est qu'on a d'après (45),

$$(47) \quad \underset{(abc)}{\mathfrak{J}} = A_{ijk\;(a)}\mu^i{}_{(b)}\mu^j{}_{(c)}\mu^k = A_{jik\;(a)}\mu^i{}_{(b)}\mu^j{}_{(c)}\mu^k = A_{jik\;(b)}\mu^j{}_{(a)}\mu^i{}_{(c)}\mu^k = \underset{(bac)}{\mathfrak{J}}.$$

Par une conséquence analogue on a de même:

$$(48) \qquad \underset{(abc)}{\mathfrak{J}} = A_{ikj\;(a)}\mu^i{}_{(b)}\mu^j{}_{(c)}\mu^k = \underset{(acb)}{\mathfrak{J}}.$$

D'après (46)–(48) l'équation (41) nous donne pour A_{ijk} en vertu de (35),

$$(49) \qquad A_{ijk} = \underset{(222)}{\mathfrak{J}}\rho^*_i\rho^*_j\rho^*_k + \underset{(223)}{\mathfrak{J}}(\rho^*_i\rho^*_j\sigma^*_k + \rho^*_i\sigma^*_j\rho^*_k + \sigma^*_i\rho^*_j\rho^*_k)$$
$$+ \underset{(233)}{\mathfrak{J}}(\rho^*_i\sigma^*_j\sigma^*_k + \sigma^*_i\rho^*_j\sigma^*_k + \sigma^*_i\sigma^*_j\rho^*_k) + \underset{(333)}{\mathfrak{J}}\sigma^*_i\sigma^*_j\sigma^*_k.$$

Supposons maintenant que

$$(50) \qquad A_{ijk}\sigma^{*k} = 0.$$

Il résulte de (50) que le tenseur A_{ijk} aura la forme:

$$(51) \qquad A_{ijk} = \underset{(222)}{\mathfrak{J}}\rho^*_i\rho^*_j\rho^*_k,$$

car d'après (42) on a

$$\underset{(ab3)}{\mathfrak{J}} = \underset{(a3b)}{\mathfrak{J}} = \underset{(3ab)}{\mathfrak{J}} = A_{ijk\;(a)}\mu^i{}_{(b)}\mu^j\sigma^{*k} = 0 \qquad (a, b = 1, 2, 3).$$

Dans un ensemble de l'espace où l'équation (50) existe, le tenseur de torsion A_{ijk} a la même forme que dans un espace à deux dimensions. Si le scalaire $\underset{(222)}{\mathfrak{J}}$ est nul le long de toutes les courbes de l'ensemble on a d'après (51) et (42),

$$A_{ijk} = 0,$$

et d'après (43),

$$g_{ik} = g_{ik}(x^1, x^2, \ldots, x^n).$$

L'ensemble est alors un espace riemannien.

De la même façon, il résulte de l'équation (40) que si le scalaire $\underset{(212)}{\mathfrak{R}}$ est nul le long de toutes les courbes de l'espace, le tenseur contracté de courbure est aussi nul:

$$R_0{}^i{}_{jk} = 0.$$

Dans ces espaces, il existe donc un parallélisme absolu des éléments linéaires [6].

Les équations (39) et (50) nous donnent une condition suffisante pour que les tenseurs $R_0{}^i{}_{jk}$ et A_{ijk} aient la même forme que dans un espace à deux dimensions, mais ces conditions ne sont pas nécessaires. Les conditions nécessaires et suffisantes ont une forme plus compliquée que les conditions données. Elles ont la forme suivante, d'après (38) et (49):

$$\underset{(213)}{\mathfrak{R}}\rho^{*i}(l_j\sigma^*_k - l_k\sigma^*_j) + \underset{(223)}{\mathfrak{R}}\rho^{*i}(\rho^*_j\sigma^*_k - \rho^*_k\sigma^*_j) + \underset{(312)}{\mathfrak{R}}\sigma^{*i}(l_j\rho^*_k - l_k\rho^*_j) \\ + \underset{(313)}{\mathfrak{R}}\sigma^{*i}(l_j\sigma^*_k - l_k\sigma^*_j) + \underset{(323)}{\mathfrak{R}}\sigma^{*i}(\rho^*_j\sigma^*_k - \rho^*_k\sigma^*_j) = 0, \tag{52}$$

$$\underset{(223)}{\mathfrak{J}}(\rho^*_i\rho^*_j\sigma^*_k + \rho^*_i\sigma^*_j\rho^*_k + \sigma^*_i\rho^*_j\rho^*_k) + \underset{(233)}{\mathfrak{J}}(\rho^*_i\sigma^*_j\sigma^*_k + \sigma^*_i\rho^*_j\sigma^*_k + \sigma^*_i\sigma^*_j\rho^*_k) \\ + \underset{(333)}{\mathfrak{J}}\sigma^*_i\sigma^*_j\sigma^*_k = 0. \tag{53}$$

L'équation (52) résulte de l'équation (39) et le tenseur de courbure aura alors la forme (40). De même, de l'équation (50) nous aurons (53). Le tenseur de torsion A_{ijk} aura alors la forme (51). Mais on ne peut conclure de l'existence des équations (52) et (53) à celle de (39) et de (50).

Nous voulons encore remarquer que tous les invariants et tous les tenseurs considérés dépendent de la courbe (8) (cf. l'introduction), car le vecteur ρ_i, ou ρ^*_i, est une fonction de x''^i (cf. les équations (11) et (29a)). Ces invariants et ces tenseurs ne caractérisent l'espace que le long de la courbe (8). Si l'élément fondamental de l'espace était l'élément linéaire d'ordre deux (x, x', x''), comme p.e. dans un espace de Kawaguchi, nous pensons qu'on pourrait obtenir des invariants de l'espace par une considération analogue, car le trièdre l^i, ρ^{*i}, σ^{*i} dépend de l'élément fondamental (x, x', x'') et le trièdre sera défini dans tous les éléments fondamentaux de l'espace, et non pas seulement le long de la courbe (8). En ce cas la fonction fondamentale a la forme:

$$ds = F(x, x', x'')dt$$

et naturellement l^i, ρ^i, σ^i ont une autre forme, que dans l'espace dont l'élément fondamental est (x, x'); p.e.

$$\rho_i = F_{x^i} - \frac{d}{dt}F_{x'^i} + \frac{d^2}{dt^2}F_{x''^i}.$$

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A CLOSURE CRITERION FOR ORTHOGONAL FUNCTIONS

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1. Introduction. In this paper we give a simple, necessary, and sufficient condition for a sequence of orthogonal functions to be closed in L_2. In theory the question of closure is reduced to the evaluation of certain integrals and the summation of an infinite series whose terms depend only upon the index n. Our principal result is

THEOREM I. *Let $p(t)$ be a function whose zeros and discontinuities have Jordan content zero, such that for each $x \in (a, b)$, $p(t) \in L_2$ on* $\min(c, x) < t < \max(c, x)$, *where $a < c < b$. (a, b, and c may be infinite.) Let $w(x)$ be a measurable function almost everywhere finite and positive, and such that*

$$w(x)\int_c^x |p(t)|^2 dt \in L_1$$

on (a, b). Then for any family of functions $\{\phi_n\}$ orthogonal and normal on (a, b).

$$\sum_{n=1}^{\infty} \int_a^b \left| \int_c^x p(t)\phi_n(t)dt \right|^2 w(x)dx \leq \int_a^b \left| \int_c^x |p(t)|^2 dt \right| w(x)dx, \tag{1.1}$$

where equality holds if and only if $\{\phi_n\}$ is closed in L_2 on (a, b).

The insertion of the functions $p(t)$ and $w(x)$ serves two purposes. First, it enables us to deal with the case where the interval (a, b) is infinite, and second, proper choice of these functions greatly facilitates the calculation of the integrals involved and the summation of the series on the left side of (1.1).

Several special cases of Theorem I were published by Dalzell [**2**, **3**]. Inasmuch as Dalzell's results were stated only for special cases and under rather stringent hypotheses, it seemed desirable to publish the criterion in the above general form in which it was rediscovered by the author.

Actually Theorem I is a special case of a more general result. Before stating this result, it is necessary to attend to certain matters of notation. Note that we distinguish carefully between open intervals (a, b) and closed intervals $[a, b]$. The distinction is necessary in order that the statements and proofs should be valid for both finite and infinite intervals. We use $|a, b|$ to denote the open interval $\min(a, b) < x < \max(a, b)$, while $\chi_E(x)$ is the characteristic function of the set E and $\|f\|$ is the L_2 norm of f.

Definition 1.1. $\mathfrak{W}_{a,b}$ is the class of measurable functions $w(x)$ which are positive and finite almost everywhere on (a, b).

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Definition 1.2. $\mathfrak{P}_{a,b}^c$, where $c \in [a, b]$, is the class of measurable functions $p(t)$ on (a, b) such that

A. For each $x \in (a, b)$, $p(t) \in L_2$ on $|c, x|$.

B. The class of functions of the form

$$f(t) = \sum_{k=1}^{m} c_k p(t) \chi_{(a_k, b_k)}(t); \qquad a_k, b_k \in (a, b), \tag{1.2}$$

is dense in L_2 on the interval (a, b).

We are now in a position to state the generalization of Theorem I:

THEOREM II. *Let $p(t) \in \mathfrak{P}_{a,b}^c$ and $w(x) \in \mathfrak{W}_{a,b}$ be such that*

$$w(x)\int_c^x |p(t)|^2 dt \in L_1$$

on (a, b), and let $\{\phi_n\}$ be a family of orthonormal functions on (a, b). Then

$$\sum_{n=1}^{\infty} \int_a^b \left| \int_c^x p(t)\phi_n(t)dt \right|^2 w(x)dx \leq \int_a^b \left| \int_c^x |p(t)|^2 dt \right| w(x)dx, \tag{1.3}$$

where equality holds if and only if $\{\phi_n\}$ is closed[1] *in L_2 on (a, b).*

The fact that condition B of Definition 1.2 is not nearly as restrictive as it appears is a consequence of the next theorem:

THEOREM III. *Let $p(t)$ be a function whose discontinuities and zeros are of Jordan content zero. Then the class of functions of the form*

$$f(t) = \sum_{k=1}^{m} c_k p(t) \chi_{(a_k, b_k)}(t); \qquad a_k, b_k \in (a, b),$$

is dense in L_2 on the interval (a, b).

Theorem I is, of course, an immediate corollary of Theorems II and III. We devote §§2 and 3 to the proofs of these theorems, while in §4 we apply Theorem I to establish the closure of the Hermite functions. For further applications see Dalzell [**2**; **3**], where the method is used to prove closure of the trigonometric, Legendre, Jacobi, and Laguerre functions, and also for Dini's series in the theory of Bessel functions.

2. Proof of Theorem II. We first establish the inequality (1.3). Except at most for a matter of sign,

$$\int_c^x \overline{p(t)}\ \overline{\phi_n(t)}dt$$

is the nth orthogonal coefficient of

$$\overline{p(t)}\chi_{|c,x|}(t).$$

[1]It will be seen from the proof that, if $\{\phi_n\}$ is closed in L_2 on (a, b), then the equality holds in (1.3) even without the restriction that the functions of the form (1.2) be dense in L_2 on (a, b).

By Bessel's inequality we have

$$(2.1)\qquad \sum_{n=1}^{\infty}\left|\int_c^x \overline{p(t)}\ \overline{\phi_n(t)}dt\right|^2 < \int_a^b\left|\overline{p(t)}\chi_{|c,x|}(t)\right|^2 dt = \left|\int_c^x |\overline{p(t)}|^2 dt\right|,$$

whence

$$(2.2)\qquad \sum_{n=1}^{\infty}\left|\int_c^x \overline{p(t)}\ \overline{\phi_n(t)}dt\right|^2 w(x) < \left|\int_c^x |\overline{p(t)}|^2 dt\right| w(x).$$

Integration of (2.2) yields

$$(2.3)\qquad \sum_{n=1}^{\infty}\int_a^b\left|\int_c^x \overline{p(t)}\ \overline{\phi_n(t)}dt\right|^2 w(x)dx < \int_a^b\left|\int_c^x |\overline{p(t)}|^2 dt\right| w(x)dx,$$

which is equivalent to (1.3).

In case $\{\phi_n\}$ is closed in L_2 on (a, b), the equalities hold in (2.1), (2.2), and (2.3), and hence equality holds in (1.3).

Suppose now that equality holds in (1.3), and thus in (2.3). Then equality must hold almost everywhere in (2.2), and hence almost everywhere in (2.1); that is, for almost all x's Parseval's equality holds for the functions

$$\overline{p(t)}\chi_{|c,x|}(t),$$

and in particular for a set of x's dense in (a, b). We conclude at once that $\{\phi_n\}$ is closed with respect to all functions of the form

$$\overline{p(t)}\chi_{|c,x|}(t);$$

thus $\{\phi_n\}$ is closed with respect to all functions of the form

$$(2.4)\qquad f(t) = \sum_{k=1}^{m} c_k\overline{p(t)}\chi_{(a_k,b_k)}(t); \qquad a_k, b_k \in (a,b).$$

But

$$p(t) \in \mathfrak{P}_{a,b}^{c}$$

implies that

$$\overline{p(t)} \in \mathfrak{P}_{a,b}^{c}.$$

Since by this remark the functions of the form (2.4) are dense in L_2, the L_2 closure of $\{\phi_n\}$ follows immediately.

3. Proof of Theorem III. As the class of step functions vanishing outside a finite interval is dense in L_2 and as every such function is a linear combination of characteristic functions of finite intervals, it will suffice to show that the characteristic function of a finite interval can be approximated arbitrarily closely in norm by functions of the form (1.2). Since the set of zeros and discontinuities of $p(t)$ can be covered by a finite number of intervals of arbitrarily small total measure, it will certainly suffice to show that we can approximate $\chi_{(\alpha,\beta)}(t)$ arbitrarily closely in L_2 norm by functions of the form (1.2), where $[\alpha,\beta]$ is a finite interval which contains no zeros or discontinuities of $p(t)$.

Let $\epsilon > 0$ be given, let m be the minimum of $|p(t)|$ on $[\alpha,\beta]$, and let $\delta > 0$ be chosen so that the oscillation of $p(t)$ on every subinterval of $[\alpha,\beta]$ whose length does not exceed δ is less than $\epsilon m(\beta - \alpha)^{-\frac{1}{2}}$. Let

$$\alpha = t_0 < t_1 < \ldots < t_n = \beta$$

be a partition of $[\alpha,\beta]$ whose norm does not exceed δ. For $t \in [t_{k-1},t_k]$ we have $|p(t_k) - p(t)| < \epsilon m(\beta - \alpha)^{-\frac{1}{2}}$, whence

$$(3.1) \qquad \left|1 - \frac{p(t)}{p(t_k)}\right| < \epsilon(\beta - \alpha)^{-\frac{1}{2}}, \qquad t \in [t_{k-1}, t_k].$$

Define

$$(3.2) \qquad f(t) = \sum_{k=1}^{n} \frac{p(t)}{p(t_k)} \chi_{(t_{k-1},t_k)}(t).$$

We see from (3.1) and (3.2) that, for $t \in (t_{k-1},t_k)$,

$$(3.3) \qquad \left|\chi_{(\alpha,\beta)}(t) - f(t)\right| = \left|1 - \frac{p(t)}{p(t_k)}\right| < \epsilon(\beta - \alpha)^{-\frac{1}{2}},$$

while both $f(t)$ and $\chi_{(\alpha,\beta)}(t)$ vanish outside of (α,β). From this last remark and (3.3) we have $\|\chi_{(\alpha,\beta)} - f\| < \epsilon$, as was to be proved.

4. Closure of the Hermite functions. The normalized Hermite functions $\{\phi_n\}$ are defined as

$$(4.1) \qquad \phi_n(x) = (\pi^{\frac{1}{2}} n! 2^n)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} H_n(x) \qquad (n = 0,1,2,\ldots),$$

where the Hermite polynominals $H_n(x)$ are given by

$$(4.2) \qquad H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \qquad (n = 0,1,2,\ldots).$$

We will use the facts that [**1**, pp. 77-79; **4**, pp. 143-144]

$$(4.3) \qquad H_{2n}(0) = (-1)^n \frac{(2n)!}{n!}, \quad H_{2n+1}(0) = 0 \qquad (n = 0,1,2,\ldots)$$

and

$$(4.4) \qquad H_n'(x) = 2nH_{n-1}(x) \qquad (n = 1,2,\ldots).$$

To establish the closure of the Hermite functions, we choose $p(t) = \exp\{\frac{1}{2}t^2\}$, $w(x) = \exp\{-2x^2\}$. According to Theorem I, the closure is equivalent to the equality

$$\sum_{n=0}^{\infty} \frac{1}{\pi^{\frac{1}{2}} n! 2^n} \int_{-\infty}^{\infty} \left[\int_0^x H_n(t)dt\right]^2 e^{-2x^2} dx = 2\int_0^{\infty} \left[\int_0^x e^{t^2} dt\right] e^{-2x^2} dx.$$

A transformation to polar coordinates and an elementary integration yield

$$2\int_0^{\infty} \left[\int_0^x e^{y^2} dy\right] e^{-2x^2} dx = 2^{-\frac{1}{2}} \log(1 + 2^{\frac{1}{2}}),$$

so that if we set

$$J_n = (2n)^2 \int_{-\infty}^{\infty} \left[\int_0^x H_{n-1}(t)\,dt \right]^2 e^{-2x^2}\,dx \qquad (n = 1, 2, \ldots),$$

it remains only to show that

$$\sum_{n=0}^{\infty} \frac{J_{n+1}}{\pi^{\frac{1}{2}} n! 2^n (2n+2)^2} = 2^{-\frac{1}{2}} \log (1 + 2^{\frac{1}{2}}). \tag{4.5}$$

Use of (4.4) yields

$$J_n = \int_{-\infty}^{\infty} H_n^2(x) e^{-2x^2}\,dx - 2H_n(0) \int_{-\infty}^{\infty} H_n(x) e^{-2x^2}\,dx + H_n^2(0) \int_{-\infty}^{\infty} e^{-2x^2}\,dx.$$

Upon integrating by parts n times and using (4.2) we obtain

$$\int_{-\infty}^{\infty} H_n^2(x) e^{-2x^2}\,dx = \int_{-\infty}^{\infty} \left[\frac{d^n}{dx^n} e^{-x^2} \right]^2 dx = (-1)^n \int_{-\infty}^{\infty} H_{2n}(x) e^{-2x^2}\,dx,$$

from which

$$J_n = (-1)^n I_{2n} - 2H_n(0) I_n + \left(\tfrac{1}{2}\pi\right)^{\frac{1}{2}} H_n^2(0), \tag{4.6}$$

where we have written

$$I_n = \int_{-\infty}^{\infty} H_n(x) e^{-2x^2}\,dx.$$

We have

$$\begin{aligned} I_{2n} &= \int_{-\infty}^{\infty} e^{-x^2} \frac{d^{2n}}{dx^{2n}} e^{-x^2}\,dx = \left[\frac{d^{2n}}{dt^{2n}} \int_{-\infty}^{\infty} e^{-x^2} e^{-(x+t)^2}\,dx \right]_{t=0} \\ &= \left[\left(\tfrac{1}{2}\pi\right)^{\frac{1}{2}} \frac{d^{2n}}{dt^{2n}} e^{-t^2/2} \right]_{t=0} = \left(\tfrac{1}{2}\pi\right)^{\frac{1}{2}} \frac{(-1)^n (2n)!}{n! 2^n}, \end{aligned} \tag{4.7}$$

while, as $H_{2n+1}(x)$ is an odd function,

$$I_{2n+1} = 0. \tag{4.8}$$

Use of (4.3), (4.6), (4.7), and (4.8) yields after some reductions

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{J_{n+1}}{\pi^{\frac{1}{2}} n! 2^n (2n+2)^2} \\ &= 2^{-3/2} \left\{ \frac{3}{2} \sum_{n=1}^{\infty} \frac{(1/2)(3/2)\ldots(n-\frac{1}{2})}{(n!)n} - \sum_{n=1}^{\infty} \frac{(1/2)(3/2)\ldots(n-\frac{1}{2})}{(n!)n} \left(\tfrac{1}{2}\right)^n \right\} \\ &= 2^{-3/2} \left\{ \frac{3}{2} K(1) - K\left(\tfrac{1}{2}\right) \right\}, \end{aligned} \tag{4.9}$$

where we have set

$$K(a) = \sum_{n=1}^{\infty} \frac{(1/2)(3/2)\ldots(n-\frac{1}{2})}{(n!)n} a^n.$$

The series defining $K(\alpha)$ converges absolutely for $|\alpha| < 1$, and for $|\alpha| < 1$,

$$K(\alpha) = \int_0^\alpha \frac{1}{x} \sum_{n=1}^{\infty} \frac{(1/2)(3/2)\dots(n-\frac{1}{2})}{n!} x^n dx \tag{4.10}$$

$$= \int_0^\alpha \frac{1}{x}\left[(1-x)^{-\frac{1}{2}} - 1\right] dx = 2\log\left[\frac{2}{1+(1-\alpha)^{\frac{1}{2}}}\right].$$

It follows from (4.10) that

$$2^{-3/2}\left\{\frac{3}{2}K(1) - \quad (\tfrac{1}{2})\right\} = 2^{-\frac{1}{2}}\log(1+2^{\frac{1}{2}}). \tag{4.11}$$

From (4.9) and (4.11) we obtain (4.5), and the closure of the Hermite functions is established.

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ON SURFACES WHOSE CANONICAL SYSTEM IS HYPERELLIPTIC

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1. Generalities. On a surface F of genus $p_g = p_a = p$ and linear genus $p^{(1)} = n + 1$ whose canonical system is irreducible, and which we shall ordinarily think of as simple and free from exceptional curves, the characteristic series of the canonical system is a semicanonical g_n^{p-2}, since the adjoint system of the canonical system is its double, so that the canonical series on a curve of the canonical system is its characteristic series doubled. This is in general free from fixed points, so that the actual grade of the canonical system is n, and the canonical model of the surface is of order n in $[p-1]$ (by which we indicate space of $p-1$ dimensions). The bicanonical model is a surface of order $4n$ in $[P_2 - 1]$ where, by a formula derived from the Riemann-Roch theorem (**9**, p.159),

$$P_2 - 1 \geqslant p_a + p^{(1)} - 1 = n + p.$$

Equality will hold in general, and we shall shortly see that it holds in all cases we are going to discuss.

On a hyperelliptic curve of genus $n + 1$ however, every semicanonical g_n^{p-2} consists of $p - 2$ variable pairs of the unique g_2^1 on the curve together with $n - 2p + 4$ fixed points, which are a subset of the $2n + 4$ jacobian points of the g_2^1; for since any two sets of the series together form a canonical set, consisting of $p - 1$ pairs of the g_2^1, the variable part of the series must consist of whole pairs of g_2^1 and each fixed point must be a half pair, i.e., a jacobian point. As an obvious corollary, $n \geqslant 2p - 4$, that is,

$$p^{(1)} \geqslant 2p - 3,$$

which is a classical formula (**9**, p.294).

Hence if the general curve of the canonical system on F is irreducible and hyperelliptic, the canonical system has $n - 2p + 4$ unassigned base points at simple points of F, and its actual grade is $2p - 4$. As the projective model of g_n^{p-2} is a double normal rational curve of order $p - 2$, the canonical model of the surface is a double surface of order $p - 2$ in $[p - 1]$ with rational hyperplane sections, i.e.,[1] either a normal rational ruled surface or (for $p = 6$ only) the Veronese surface V_2^4. We shall denote the ruled surface by R_2^{p-2}; for $p = 3$, of course, it is a plane, and for $p = 2$ it can hardly be held to exist. On V_2^4 or R_2^{p-2} the $n - 2p + 4$ base points P_i of the canonical system appear as ex-

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[1]For standard properties of rational surfaces reference may be made to (**3**); for the present result, pp. 271 ff., 298.

ceptional lines λ_i ($i = 1, \ldots, n - 2p + 4$) which, moreover, are constituents of the branch curve of the double surface, since each P_i is a jacobian point of the g_2^1 on each curve of the canonical system, i.e., contributes a branch point to the general hyperplane section of the double surface; and as the general hyperplane section has $2n + 4$ branch points altogether, there is a residual branch curve[2] of order $n + 2p$. We note as an obvious corollary that if for $p = 6$ the canonical model is the double Veronese surface, we must have $n = 8$, i.e., $p^{(1)} = 9$, its lowest value for $p = 6$, since there are no lines on the surface; in this case the branch curve is of order 20, and the surface is equivalent to a double plane with general branch curve of order 10 (see exceptional case (i) below).

The bicanonical model is a double rational surface Φ^{2n} on which the base points P_i appear as points, since the bicanonical system traces on each curve of the canonical system its canonical series, compounded with g_2^1, and has no base points at P_i. The projection of Φ^{2n} from these points is a surface Ψ^{4p-8}, projective model of the system of all quadric sections of R_2^{p-2} (or of all conics in the plane for $p = 2$). The ambient space of this latter is $[3p - 4]$, as the freedom of quadrics in the ambient of R_2^{p-2} is $\frac{1}{2}(p-1)(p+2)$, and R_2^{p-2} is itself the intersection of $\frac{1}{2}(p-2)(p-3)$ linearly independent quadrics, its equations being the vanishing of all quadratic minors in a matrix of 2 rows and $p - 2$ columns whose elements are linear in the coordinates. The projections of the points P_i are the images of the lines λ_i, i.e., they are conics S_i, so that the points P_i are isolated branch points at conical nodes of Φ^{2n}; on F, all curves of the bicanonical system that pass through a point P_i have a double point there. Φ^{2n} also has a branch curve of order $2n + 4p$ not passing through the points P_i, which projects into a curve of the same order on Ψ^{4p-8}, image of f^{n+2p}, the residual branch curve of the double R_2^{p-2}. Since the bicanonical system is of genus[3] $3n + 1$, and consists of doubled hyperplane sections of Φ^{2n} with $2n + 4p$ branch points, the section genus π of Φ^{2n} is given by

$$3n = 2(\pi - 1) + \tfrac{1}{2}(2n + 4p),$$

that is,

$$\pi = n - p + 1.$$

The general hyperplane section of Φ^{2n} is a curve of order $2n$ and genus π, non-special since $2n > 2\pi - 2$, and its ambient is therefore of dimensions at most $2n - \pi = n + p - 1$; but we have seen that the ambient of Φ^{2n} is of dimensions at least $n + p$, consequently Φ^{2n} is in $[n + p]$ precisely. A further consequence of this is that, as the difference in dimensions between the ambients of Φ^{2n} and Ψ^{4p-8} is just $n - 2p + 4$, the $n - 2p + 4$ points P_i, from which the former is projected into the latter, are all linearly independent, i.e., their join Ω is an $[n - 2p + 3]$ precisely; and further, as the difference in the order of the two

[2] The cases $n = 2p - 4$ are briefly treated by Enriques (9, p. 296). His case (II) is our exceptional case (i) ($p = 7$ is a misprint for $p = 6$).

[3] See the formulae for the genera of the bicanonical system in (9, p. 61) putting $p^{(1)} = n + 1$.

surfaces is just $2(n-2p+4)$, Ω does not meet Φ^{2n} except in the double points $P_1, \ldots, P_{n-2p+4}$.

The lines λ_i on R_2^{p-2} and conics S_i on Ψ^{4p-8} are thus of virtual grade -2 with respect to the base points of the system $|\phi|$ which represents, on either surface, the hyperplane sections of Φ^{2n}; they are fundamental to $|\phi|$, and also have no intersections outside of these base points either with each other or with the residual branch curve. They satisfy $|\phi| \equiv |\psi + \Sigma\lambda_i|$, where $|\psi|$ represents the quadric sections of R_2^{p-2}, or hyperplane sections of Ψ^{4p-8}.

An s-ple base point A of $|\phi|$ on Ψ^{4p-8} arises in projection from a curve a^s of order s on Φ^{2n}; a^s cannot have a multiple point at any P_i since its multiplicity at P_i is equal to that of S_i, an irreducible conic, in A. a^s thus passes simply through precisely s of the points P_i, say

$$P_{i_1}, \ldots, P_{i_s},$$

and the corresponding conics S_i intersect in A; the corresponding s lines λ_i meet in the image of A on R_2^{p-2}, which we may likewise call A. These s points P_i are joined to A by an $[s]$ which is properly the ambient of a^s since the latter passes through these s points and is not contained in the $[s-1]$ joining them; a^s is thus a rational curve.

If A is a simple point of Ψ^{4p-8}, that is, of R_2^{p-2}, a^s is of virtual grade -1 on Φ^{2n}, and the corresponding hyperelliptic curve on F is of virtual grade -2, with respect to the points

$$P_{i_1}, \ldots, P_{i_s},$$

and hence of virtual grade $s-2$ without regard to any base points; since the latter curve has s intersections with a general curve of the canonical system, its own canonical series is of order $2s-2$, i.e., its genus is s, and as the double a^s has branch points at

$$P_{i_1}, \ldots, P_{i_s},$$

it has $s+2$ elsewhere, i.e., a^s meets the branch curve of Φ^{2n} in $s+2$ points, and A is an $(s+2)$-ple point on the residual branch curve of R_2^{p-2} or of Ψ^{4p-8}. If A is a multiple point of Ψ^{4p-8}, it must also be multiple on R_2^{p-2}, that is, R_2^{p-2} is a cone, and A is its vertex; this special case will be considered later.

The case $p = 2$ is somewhat peculiar in that the canonical system is a pencil, its characteristic series has no variable part, there is no canonical model, and no surface Ψ^{4p-8}. We shall find it possible however to study the bicanonical models in this case also.

2. The standard case. An obviously possible arrangement of lines λ_i, and the only one possible for high values of p or n, is for them all to be generators of R_2^{p-2}, or (for $p = 3$) lines of a pencil in the double plane. In this case they must, to be of virtual grade -2, each contain two simple base points of $|\phi|$, say A_{2i-1}, A_{2i} on λ_i (in addition to the vertex A_0 of the pencil for $p = 3$, which is an $(n-2p+4 = n-2)$-ple base point). The curves of $|\phi|$ are coresidual to a

quadric section of R_2^{p-2} (conic in the plane) together with $n - 2p + 4$ generators (lines of the pencil), i.e., they are curves of order n meeting each generator in two variable points, and having the points $A_1, \ldots, A_{2n-4p+8}$ as simple base points, since each lies on just one of the lines λ_i. The residual branch curve f^{n+2p} has thus triple points at $A_1, \ldots, A_{2n-4p+8}$, and since these absorb all its intersections with the lines λ_i, it must meet each generator in six points (and for $p = 3$ have also an n-ple point at A_0). Φ^{2n} is accordingly a surface with hyperelliptic (or elliptic or rational) hyperplane sections of genus $\pi = n - p + 1$, belonging to the series studied classically by Castelnuovo (**1**; **3**, p.464), but special in having $n - 2p + 4$ conical nodes. In particular, for $n = 2p - 4$, Φ^{2n} coincides with Ψ^{4p-8} and is the projective model of all quadric sections of R_2^{p-2}, its section genus being $\pi = p - 3$ and its order $4p - 8 = 4\pi + 4$, the highest possible for surfaces with hyperelliptic sections of this genus (**9**, p. 296, Case III). For $p = 3$, $n = 2$, this gives the Veronese surface with 16-ic branch curve, corresponding to the familiar canonical double plane with octavic branch curve (**9**, p.311; p.296, Case I), and for $p = 4$, $n = 4$, the supernormal octavic del Pezzo surface (on which are no lines, and two pencils of conics) with 24-ic branch curve, corresponding to the canonical double quadric surface branching along a general sextic section (**9**, p.270).

Φ^{2n} has on it a pencil of conics, corresponding to the generators of R_2^{p-2} (or to the lines of the pencil), which trace on each hyperplane section its quadratic involution, and the pencil is accordingly unique for $p > 2$. The planes of these conics generate a rational threefold R_3^{n+p-2}, normal because the surface Φ^{2n} on it is normal and is obviously not coresidual to a hyperplane section or any part of one (the hyperplane sections of R_3 being rational ruled surfaces), and hence of order $n + p - 2$, since it is in $[n + p]$, and R_3^s is normal in $s + 2$. Φ^{2n} is the residual section of R_3^{n+p-2} by a quadric through $2p - 4$ of these planes, since every surface on R_3^s which meets the general plane in a t-ic curve is coresidual to a t-ic section plus or minus a suitable number of planes to make its order right. In the same way, since the branch curve of Φ^{2n} meets each conic of the pencil in six points, passes through none of the nodes, and is of order $2n + 4p$, it is the residual section of Φ^{2n} by a cubic through $2(n - p) = 2(\pi - 1)$ conics. If $\pi = 1$, Φ^{2n} is a del Pezzo surface, on which the pencil of conics is not unique, and the branch curve is a complete cubic section not residual to any conics; the values $n = p = 4, 3, 2$ respectively give for Φ^{2n}, the supernormal del Pezzo surface of order 8 just referred to, that of order 6 with a double point which is not base point of any pencil of conics on the surface (represented on a plane by cubics with three colinear base points) and that of order 4 with two double points, intersection of two quadrics in [4] one of which is a cone with line vertex. The first and last of these are mentioned by Enriques (**9**, pp. 270, 314); the other corresponds to a canonical double plane whose branch curve consists of a line λ together with a curve of order 9 with three triple points A_0, A_1, A_2 lying in λ. If $\pi = 0$ of course the pencil of conics is replaced by a larger system, and the branch curve is coresidual to a cubic section together with two conics. The only

possible cases are $p = 3$, $n = 2$, which gives the double Veronese surface above, with 16-ic branch curve and $p = 2$, $n = 1$, which gives a double quadric cone in [3], branching along a general quintic section and having an isolated branch point at its vertex (**9**, p. 304).

It is worth remarking that the surfaces Φ^{2n} of section genus π ($n < 2\pi + 2$) with $n - 2p + 4 = 2\pi + 2 - n$ conical nodes fall into a sequence for diminishing values of n, in which each is the projection of the preceeding one from a tangent line, i.e., is obtained from it by imposing two simple base points, say X_1, X_2 on the hyperplane sections, of which X_2 is in the neighbourhood of X_1. This gives on the projected surface a new conical node corresponding to the neighbourhood of X_1, through which pass two lines, one corresponding to the neighbourhood of X_2, and the other to that conic of the pencil on the original surface which passes through X_1, the two together forming a degenerate conic of the pencil on the new surface. At the same time, for the double surface to be bicanonical, the branch curve must have triple points at X_1, X_2, i.e., what Enriques calls a [3,3] point. In fact, applying, as we evidently can, Enriques' study (**9**, pp. 77-79) of the behaviour of the canonical and bicanonical curves at singular points of the branch curve of a double plane to a general point of any double surface, we see that if any double surface Σ has a branch curve variable in a linear system, as long as the branch curve acquires no extra singularity, the canonical and bicanonical models remain unchanged, except of course for the variation of the branch curve. But when the branch curve acquires a new [3,3] point X_1,X_2, the canonical system acquires a simple base point at X_1 and the bicanonical system a [1,1] point, i.e., simple base points at both X_1 and X_2. The new canonical model is thus the projection of the old from a point, and the new bicanonical model is the projection of the old from the tangent line $X_1 X_2$. On the canonical model the line arising from the neighbourhood of X_1 is a constituent of the branch curve, and the residual branch curve has a triple point at X_2 on this line. On the bicanonical model the node arising from the neighbourhood of X_1 is an isolated branch point, and the line arising from the neighbourhood of X_2 is not part of the branch curve but meets the branch curve in three points distinct from the node. When, as in the present case, there is a conic on Σ passing through X_1 and meeting the branch curve in six (or more generally in s) points, the line arising from this meets the branch curve in three (or $s - 3$) points, distinct from the node. There is thus unit diminution both in the genus of the surface and in its linear genus (since by the coincidence of three of its branch points in X_1 the general curve of the canonical system effectively loses two of them). To sum up, by projecting a bicanonical surface Φ^{2n} of genus p from a tangent line, we obtain a bicanonical surface $\Phi^{2(n-1)}$ of genus $p - 1$, provided that the branch curve is at the same time so specialized that the virtual difference between the branch curve and a complete cubic section of $\Phi^{2(n-1)}$ is the projection of the similarly defined system on Φ^{2n}. For, the hyperplane sections of $\Phi^{2(n-1)}$ are represented on Σ by the same system as those of Φ^{2n}, with the imposition of the [1,1] base point, and thus the cubic sections of $\Phi^{2(n-1)}$ are

represented by the same system as those of Φ^{2n}, with the imposition of a [3,3] point. Since the branch curve simultaneously acquires the same singularity, their virtual difference is unchanged.

Hitherto, we have tacitly assumed R_2^{p-2} to be the general normal rational ruled surface of this order. We may now consider the possibility of its being one of the more special types, i.e., having a directrix (curve unisecant to its generators) of lower order than in the general case, or even in the extreme case of its being a cone. For this investigation it is convenient to map R_2^{p-2} on a plane, as we always can, so that its hyperplane sections correspond to curves of order $p-1$ with a $(p-2)$-ple base point X, and $p-1$ simple base points $Y_1, \ldots, Y_{p-1}$. The generators correspond to the pencil of lines through X; thus the images of A_{2i-1}, A_{2i} (which we can conveniently indicate by the same symbols) are collinear with X. The branch curve f^{n+2p} is represented by a curve of order $n+2p+6$, with an $(n+2p)$-ple point at X, sextuple points at $Y_1, \ldots, Y_{p-1}$, and triple points at $A_1, \ldots, A_{2n-4p+8}$. With the addition of the $n-2p+4$ lines $XA_{2i-1}A_{2i}$, this gives a total branch curve for the double plane, of order $2n+10$, with a $(2n+4)$-ple point at X, sextuple points at $Y_1, \ldots, Y_{p-1}$, and quadruple points at $A_1, \ldots, A_{2n-4p+8}$. The virtual canonical system is thus[4] of order $n+2$ with $(n+1)$-ple base point at X, double base points at $Y_1, \ldots, Y_{p-1}$, and simple base points at $A_1, \ldots, A_{2n-4p+8}$. The $n-2p+4$ lines $XA_{2i-1}A_{2i}$, and the $p-1$ lines XY_i (which are pairs of coincident exceptional lines on the double plane) have negative virtual intersection numbers with this system and separate out, leaving, as we expect, the system representing the hyperplane sections of R_2^{p-2}.

The general R_2^{p-2} has (if p is odd) a single minimum directrix of order $\frac{1}{2}(p-3)$, or (if p is even) a pencil of minimum directrices of order $\frac{1}{2}(p-2)$. We can give it a minimum directrix of order $k < \frac{1}{2}(p-4)$ (which includes making the surface a cone if $k=0$) by letting all but k of the points $Y_1, \ldots, Y_{p-1}$, say $Y_{k+1}, \ldots, Y_{p-1}$, lie on a line L. If

$$n+2p+6 > 6(p-1-k)$$

i.e., if $n > 4p-12-6k$ (which, since always $n > 2p-4$, will always be the case if $k > \frac{1}{3}(p-4)$), the above argument remains valid, and the minimum directrix plays no more special role on the surface than in the general case. If $n < 4p-12-6k$ the line L separates out of the branch curve f^{n+2p+6}, giving a residual branch curve of order $n+2p+5$ with sextuple points at $Y_1, \ldots, Y_k$, and quintuple at $Y_{k+1}, \ldots, Y_{p-1}$; thus the k lines $XY_1, \ldots, XY_k$ also separate out, leaving a curve of order $n+2p-k+5$ with an $(n+2p-k)$-ple point at X, and quintuple points at $Y_1, \ldots, T_{p-1}$. In this case, moreover, one of each pair A_{2i-1}, A_{2i}, say A_{2i}, must be on L, and the residual branch curve then has triple point at A_{2i-1} and a double point at A_{2i}. On R_2^{p-2} the branch curve, besides the $n-2p+4$ exceptional generators λ_i, contains the minimum di-

[4]Using again the rules for finding the canonical system of a double plane given e.g. by Enriques (9. pp. 77-79).

rectrix as part, with residual part of order $n + 2p - k$; the intersection of each λ_i with the directrix is double on this residual branch curve, which has also a triple point elsewhere on each λ_i. This case is only possible however if

$$n + 2p - k + 5 > 5(p - k - 1) + 2(n - 2p + 4),$$

i.e., if $n < p + 4k + 2$, otherwise L will separate out a second time leaving an effective total branch curve of order $2n + 8$ with a $(2n + 4)$-ple point at X, so that the canonical system is compounded with the pencil of lines through X. Thus if $n < 4p - 6k - 13$ we must have also $n < p + 4k + 2$; in particular, if

$$2p - 4 < p + 4k + 2 < 4p - 6k - 14,$$

i.e., if

$$\tfrac{1}{4}(p - 6) < k < \tfrac{1}{10}(3p - 16),$$

there is a gap in the values of n for which the double R_2^{p-2} with minimum directrix of order k can be a canonical surface of genus p and linear genus $n + 1$. For

$$2p - 4 < n < p + 4k + 2$$

and for

$$n > 4p - 6k - 12$$

the surface exists (the minimum directrix being a part of the branch curve in the former case), but for

$$p + 4k + 3 < n < 4p - 6k - 13$$

there is no such surface.

In particular let us consider the case $k = 0$, i.e., that in which R_2^{p-2} is a cone.

For $p = 4$ every value of $n > 2p - 4 = 4$ is possible, the residual branch curve of order $n + 8$ passing $n - 4$ times through the vertex, and meeting each generator in six points, so that the total branch curve of order $2n + 4$ passes $2n - 8$ times through the vertex, and a general curve passing through the vertex does not branch there.

For $p = 5$, again every value of $n > 2p - 4$ is possible, but for $n = 6, 7$ the vertex is a branch point on the general curve through it; for $n = 6$ the branch curve of order 16 passes simply through the vertex and meets each generator elsewhere in 5 points; for $n = 7$ there is a single branch generator, and the residual branch curve of order 17 passes twice through the vertex, its two branches touching the branch generator (in which it has elsewhere a triple point) and meets the general generator in five variable points; for $n > 8$ there are $n - 6$ branch generators, and the residual branch curve of order $n + 10$ passes $n - 8$ times through the vertex and meets each generator in six further points.

For $p = 6$ we have the gap referred to above. For $n = 8$ the branch curve of order 20 is a quintic section, and there is an isolated branch point at the vertex; for $n = 9, 10, 11$ the surface does not exist; while for $n > 12$ the general curve through the vertex does not branch there, as there are $n - 8$ branch generators, and the residual branch curve of order $n + 12$ has $n - 12$ branches

through the vertex; as in the general case, it meets each generator in six points, and has two triple points (distinct from each other and from the vertex) in each of the branch generators λ_i. For higher values of p, the canonical surface can only be a cone if $n > 4p - 12$.

3. The exceptional cases. It is clear that the standard case just considered, in which the lines λ_i are generators of R_2^{p-2}, is the only one possible for $p > 7$; since the general R_2^{p-2} has no line on it except its generators, and even if R_2^{p-2} is specialized to have a directrix line, this is of grade $4 - p$, and cannot, by the imposition of any base points, be made of grade -2 (as it must be to be a line λ_i) unless $4 - p > -2$, that is, $p < 6$. We shall consider the possible cases for values of p in descending order.

(i) $p = 6$, $n = 8$. In this case, as we have seen, and in this case only, the canonical model may be a double Veronese surface instead of a ruled surface R_2^{p-2}. The branch curve f^{20} is its section by a general quintic, and corresponds to a general curve of order 10 in the standard plane mapping of the Veronese surface (**9**, p. 296, case II). Φ^{16} is thus the projective model of all quartics in the plane, and its branch curve f^{40} is its residual section by a cubic through a rational curve of order 8, image of a conic in the plane.

If now R_2^4 has a directrix line, this is already of grade -2, and needs no points A_i in it to make it so. Thus if this directrix is a line λ it is the only one, since if there were any other it could only be a generator, and its intersection with the directrix would have to be a base point A_i. Thus we have the single case:

(ii) $p = 6$, $n = 9$. R_2^4 has a directrix line which is the unique branch line λ. The residual branch curve f^{21} does not meet λ, and hence meets each generator in seven points, and is the residual section of R_2^4 by a septimic through seven generators. R_2^4 is the projective model of the complete system of rational cubics on a quadric cone in [3], and Φ^{18} is accordingly the projective model of twice this system together with the neighbourhood of the vertex (which is the image of λ), i.e., of the complete system of cubic sections of the cone. The branch curve f^{42} is the image of a septimic section of the cone, and is the residual section of Φ^{18} by a cubic through an elliptic 12-ic curve, image of a quadric section of the cone.

Turning to the case $p = 5$, we have to consider the possibility of the directrix line of R_2^3 being a line λ_i; since its grade, without base points, is -1, it must have one base point A in it to reduce the grade to -2, and as the intersection of any two lines λ_i, λ_j must be a base point, any other line λ_j can only be the generator through A. We thus have two cases:

(iii) $p = 5$, $n = 7$. The unique branch line λ is the directrix of R_2^3, and contains the unique simple base point A of $|\phi|$, which is also a triple point of the residual branch curve f^{17}. $|\phi|$ consists of curves of order 7 meeting each generator in three points and λ only in A, i.e., is the complete system of residual sections of R_2^3 by cubics through two generators and A; similarly f^{17} meets λ only in the triple point A, and each generator consequently in 7 points, and is thus the residual section by a septimic through four generators. If R_2^3 is mapped on a

plane by conics with a simple base point X, A corresponds to a point in the neighbourhood of X, so that $|\phi|$ corresponds to the complete system of quartics with a [1,1] point in X and A, and f^{17} to a curve of order 10 with a [3,3] point there.

(iv) $p = 5$, $n = 8$. λ_1 is the directrix and λ_2 a generator of R_2^3; their intersection A_1 is a double base point of $|\phi|$, and there is also a simple base point A_2 on λ_2. $|\phi|$ consists of octavic curves trisecant to the generators; that is, $|\phi|$ is the complete system of residual sections of R_2^3 by cubics through one generator and A_2, and touching the surface in A_1. The branch curve meets each generator in 7 points, and has a quadruple point in A_1 and a triple point in A_2. If R_2^3 is projected into a quadric cone in [3] from A_1, f^{18} becomes a septimic section with a [3,3] point at the images of (λ_2, A_2), and $|\phi|$ the complete system of cubic sections of the cone with a [1,1] base point there.

For $p = 4$ we have to consider the possibility that the lines λ_i include generators of both systems of R_2^2; in this case they must be not more than two in each system, since the intersections of any one with all those of the other system must be included in the two base points A_i which are needed on the generator to reduce its grade to -2. There are thus the following three cases:

(v) $p = 4$, $n = 6$. λ_1, λ_2 are generators of opposite systems of R_2^2; their intersection A_0 is a double base point of $|\phi|$, which consists of cubic sections of R_2^2, and has also two simple base points A_1 on λ_1 and A_2 on λ_2. f^{14} is a complete septimic section with a quadruple point at A_0 and triple points at A_1, A_2. $|\phi|$ and f^{14} project from A_0 into the complete system of plane quartics with [1,1] base points at the images of (λ_1, A_1) and (λ_2, A_2), and a curve of order 10 with [3,3] points at the same points.

(vi) $p = 4$, $n = 7$. λ_1, λ_2 belong to one system of generators and λ_3 to the other; $|\phi|$ meets each generator of the former system in 3 and of the latter in 4 points, and has two double base points A_i at the intersection of λ_i, λ_3, and two simple base points A_{i+2} lying on λ_i $(i = 1, 2)$. f^{15} meets generators of the former system in 7 and of the latter in 8 points, and has quadruple points in A_1,A_2 and triple points in A_3,A_4. The projective model of rational cubics (bisecant to the latter system) through A_1,A_2 is a quadric cone whose vertex is the image of λ_3; on this $|\phi|$ appears as the complete system of cubic sections with [1,1] base points in the images of (λ_1, A_3) and (λ_2, A_4), and f^{15} as a septimic section with [3.3] points in the same places.

(vii) $p = 4$, $n = 8$. λ_1, λ_2 belong to one system and λ_3,λ_4 to the other. $|\phi|$ consists of quartic sections, with double base points at the four points $A_1, \ldots, A_4$ of intersection of λ_1,λ_2 with λ_3,λ_4. f^{16} is an octavic section with quadruple points in $A_1, \ldots, A_4$. The projective model of the elliptic quartice through $A_1, \ldots, A_4$ is a four-nodal Segre (or quartic del Pezzo) surface, intersection of two quadric cones with line vertices in [4], on which $|\phi|$ appears as the complete system of quadric sections and f^{16} as a quartic section. f^{32} is thus a complete quadric section of Φ^{16}.

In the case $p = 3$ the canonical model is a double plane, and Ψ^4 is the Veronese surface. The lines $\lambda_1, \ldots, \lambda_{n-2}$ and base points $A_1, \ldots, A_r$ satisfy the conditions that precisely three points lie on each line, at least one line passes through each of the points, and the intersection of every two of the lines is a point of the set. It is easily verified that, apart from the standard case in which all the lines belong to one pencil, the only possibilities are the following:

(viii) $p = 3$, $n = 5$, $r = 6$. $\lambda_1, \lambda_2, \lambda_3$ are the sides and A_1, A_2, A_3 the vertices of a triangle, and A_{i+3} lies in λ_i only ($i = 1,2,3$). The system $|\phi|$ and residual branch curve are

$$\phi^5(A_1^2, A_2^2, A_3^2, A_4^1, A_5^1, A_6^1);\quad f^{11}(A_1^4, A_2^4, A_3^4, A_4^3, A_5^3, A_6^3).$$

The quadratic transformation based on A_1, A_2, A_3 transforms these into quartics with three [1,1] base points at the images of (λ_i, A_{i+3}), and f into a curve of order 10 with [3,3] points at the same points.

(ix) $p = 3$, $n = 6$, $r = 7$. $\lambda_2, \lambda_3, \lambda_4$ meet in A_1, λ_1 meets λ_i in A_i and A_{i+3} is on λ_i only ($i = 2,3,4$). $|\phi|$ and the residual branch curve are

$$\phi^6(A_1^3, A_2^2, A_3^2, A_4^2, A_5^1, A_6^1, A_7^1);\quad f^{12}(A_1^5, A_2^4, A_3^4, A_4^4, A_5^3, A_6^3, A_7^3).$$

The projective model of the cubics with base points $(A_1^2, A_2^1, A_3^1, A_4^1)$ is a quadric cone whose vertex corresponds to λ_1, on which $|\phi|$ appears as the complete system of cubic sections with [1,1] base points at the images of (λ_i, A_{i+3}), and f as a septimic section with [3,3] points at these same points.

(x) $p = 3$, $n = 6$, $r = 6$. $\lambda_1, \ldots, \lambda_4$ are the sides and $A_1, \ldots, A_6$ the vertices of a complete quadrilateral. $|\phi|$ and the residual branch curve are

$$\phi^6(A_1^2, \ldots, A_6^2);\quad f^{12}(A_1^4, \ldots, A_6^4).$$

The projective model of the cubics with base points $(A_1^1, \ldots, A_6^1)$ is a four nodal cubic surface, on which $|\phi|$ appears as the complete system of quadric sections, and f as a complete quartic section. The branch curve f^{24} is thus a complete quadric section of Φ^{12}.

(xi) $p = 3$, $n = 7$, $r = 7$. $\lambda_1, \ldots, \lambda_4$ are the sides and $A_1, \ldots, A_6$ the vertices of a complete quadrilateral; λ_5 joins the opposite vertices A_1A_2, and A_7 lies in λ_5 only. $|\phi|$ and the residual branch curve are

$$\phi^7(A_1^3, A_2^3, A_3^2, A_4^2, A_5^2, A_6^2, A_7^1);\quad f^{13}(A_1^5, A_2^5, A_3^4, A_4^4, A_5^4, A_6^4, A_7^3).$$

The projective model of the quartics with base points $(A_1^2, A_2^2, A_3^1, A_4^1, A_5^1, A_6^1)$ is a four nodal Segre surface in [4] on which $|\phi|$ appears as the complete system of quadric sections with a [1,1] base point at the image of (λ_5, A_7), and f as a complete quartic section with a [3,3] point at the same point.

(xii) $p = 3$, $n = 8$, $r = 7$. $\lambda_1, \ldots, \lambda_6$ are the sides, $A_1, \ldots, A_4$ the vertices, and A_5, A_6, A_7 the diagonal points of a complete quadrangle. $|\phi|$ and the residual branch curve are

$$\phi^8(A_1^3, A_2^3, A_3^3, A_4^3, A_5^2, A_6^2, A_7^2);\quad f^{14}(A_1^5, A_2^5, A_3^5, A_4^5, A_5^4, A_6^4, A_7^4).$$

The quadratic transformation based on A_2, A_3, A_4 changes these into the system of septimics with a triple point A and three [2,2] points, the fixed tangents in which are concurrent in the triple base point; and into a 13-ic with quintuple point at A and [4,4] points at the other base points. There is a pencil of rational quartics on Φ^{16} corresponding to conics through $A_1, \ldots, A_4$ in the first plane mapping, and to lines through A in the second, and the branch curve of Φ^{16} is the residual section by a quadric through one curve of this pencil.

It is to be remarked that these twelve bicanonical models fall into series, like those in the standard case, each member of the series being obtained from the previous one by imposing a [3,3] point on the branch curve, and correspondingly a simple base point on the canonical and a [1,1] base point on the bicanonical system, so that the canonical models are obtained by repeated projection from simple points and the bicanonical by repeated projection from tangents. These are:

$\pi = 3, p \leqslant 6, n \leqslant 8$: Projective model of the plane system of quartics with $6 - p = 8 - n$ [1,1] base points, the branch curve being mapped by a 10-ic curve with [3,3] points at the same points. $|3\phi - f|$ is mapped by all conics in the plane.

$\pi = 4, p \leqslant 6, n \leqslant 9$: Projective model of the cubic sections of a quadric cone (or of plane sextics with a [3,3] base point) having $6 - p = 9 - n$ [1,1] base points. The branch curve is mapped on the cone by a septimic section (or on the plane by a 14-ic with [7,7] point) having [3,3] points at the same points. $|3\phi - f|$ is mapped by all quadric sections of the cone (or by quartics with a [2,2] base point).

$\pi = 4, p \leqslant 3, n \leqslant 6$: Projective model of the quadric sections of a four-nodal cubic surface, or of plane sextics with six double base points which are the vertices of a complete quadrilateral. We may add $3 - p = 6 - n$ [1,1] base points to cover the case $p = 2$ which we have not yet considered. The branch curve is mapped by a quartic section of the cubic, or by a 12-ic curve with quadruple points at the vertices of the quadrilateral (and [3,3] points at any [1,1] base points of $|\phi|$). $|3\phi - f|$ is mapped by all quadric sections of the cubic, i.e., by $|\phi|$ without the [1,1] base points.

$\pi = 5, p \leqslant 4, n \leqslant 8$: Projective model of the quadric sections of the four nodal Segre (quartic del Pezzo) surface, or of the plane octavics with quartic base points at two opposite vertices and double base points at the remaining vertices of a complete quadrilateral, and $4 - p = 8 - n$ [1,1] base points. The branch curve is mapped by a quartic section of the Segre surface with [3,3] points at the [1,1] base points of $|\phi|$, i.e., $|3\phi - f|$ is mapped by quadric sections of the Segre surface, or by $|\phi|$ without its [1,1] points.

$\pi = 6, p \leqslant 3, n \leqslant 8$: Projective model of the plane system of septimic curves with one triple and three [2,2] base points, the fixed tangents at the latter all passing through the former; and of course $3 - p = 8 - n$ [1,1] points. The branch curve is mapped by a 13-ic curve with a quintuple point, and three [4,4] points, and $3 - p = 8 - n$ [3,3] points, at the base points of $|\phi|$; so that $|3\phi - f|$

is the system $|k + \phi|$ without its [1,1] base points, where $|k|$ is the pencil of quartics mapped by lines through the triple base point of $|\phi|$.

As the last surface is less known than the others we may remark that (for $p = 3$, $n = 8$) it is a surface of order 16 in [13], and the ambient [4]s of the quartics $|k|$ generate a locus R_5^9. Three of the quartics consist of a conic repeated, the tangent planes at whose points lie in the corresponding [4] and which passes through two of the six nodes. The system $|\phi|$ can be represented as the residual sections of a sextic surface with hyperplane sections of genus 2, by quadrics through one conic; the sextic surface being special in having six nodes lying by pairs on three torsal lines which, counted twice, form conics of the unique pencil of conics on the surface.

4. The surfaces of genus 2. The case $p = 2$ presents some difficulty, on account of the absence of the canonical model and of the surface Ψ^{4p-8}. The bicanonical model however must still be a double rational surface Φ^{2n} in $[n + 2]$, having n isolated branch points at conical nodes, which are base points of the canonical pencil. The canonical pencil consists of normal rational curves of order n, whose ambient $[n]$s clearly generate a quadric cone Γ_{n+1}^2 with $[n - 1]$ vertex Ω_{n-1}, which is the join of the n nodes, since any two curves of the pencil together form a hyperplane section of the surface. The hyperplane sections of Φ^{2n} have genus $\pi = n - 1$.

Now there is clearly a surface Φ^{2n} to be obtained from each of those obtained in the case $p = 3$, by imposing one further [1,1] base point on $|\phi|$ and the corresponding [3,3] point on the branch curve. In particular the standard case leads to a surface Φ^{2n}, intersection of a rational normal three-fold R_3^n generated by ∞^1 planes with the quadric cone Γ_{n+1}^2, the nodes being the intersection of R_3^n with the vertex Ω_{n-1}. We may also list immediately the exceptional cases (xiii), . . . , (xvii), obtained by imposing a [3,3] point on the branch curve of each of the double planes (viii), . . . , (xii), i.e., the representatives for $p = 2$ of the five sequences of exceptional cases already formed.

It is not immediately obvious what further cases we may expect to find. Let us suppose however that Φ^{2n} is mapped on a plane by a linear system $|\phi|$, of grade $\nu = 2n$, genus $\pi = n - 1$, and freedom $\rho = n + 2$, having ϵ base points $X_1, \ldots, X_\epsilon$, of multiplicities $i_1, \ldots, i_\epsilon$ respectively, the curves of $|\phi|$ being of order m. It is clear that we need consider only systems none of whose base points are simple since, whatever the branch curve of a multiple plane, the bicanonical system cannot have an isolated simple base point; and if it has a [1,1] point corresponding to a [3,3] point of the branch curve, the surface belongs to one of the series already enumerated for $p > 3$. Thus all the base points of $|\phi|$ are likewise base points of the adjoint system $|\phi'|$, and the conditions imposed by them are of course independent for $|\phi'|$. We need also consider only systems for which $\pi > 3$, i.e., $n > 4$, since if the curves of $|\phi|$ are rational, elliptic, or hyperelliptic, the surface is included in the standard case. The virtual intersection number ξ of $|\phi|$ with the system of cubics through all the base points is

given by

$$\xi = \nu - 2\pi + 2 = 4.$$

Now I have shewn elsewhere (5) that the grade ν' and genus π' of $|\phi'|$ and the quantity $\xi' = \nu' - 2\pi' + 2$ satisfy

$$\pi - \pi' = \xi', \quad \nu - \nu' = \xi + \xi', \quad \xi - \xi' = 9 - \epsilon.$$

From the last of these we have $\xi' = \epsilon - 5$, and putting this value with $\nu = 2n$, $\pi = n - 4$, $\xi = 4$ into the other two relations we have

$$\nu' = 2n - \epsilon + 1, \quad \pi' = n - \epsilon + 4.$$

The freedom ρ' of $|\phi'|$ is of course

$$\rho' = \pi - 1 = n - 2,$$

and consequently satisfies

$$\rho' - \pi' - \epsilon + 6 = 0.$$

It need hardly be said that these characters of $|\phi'|$ are calculated with respect only to those base points which are imposed by those of $|\phi|$. If $|\phi'|$ happens to have any other base points they are to be regarded simply as fixed points of its characteristic series of order ν'.

Now let us consider the order m of a linear system of curves, and its multiplicities $i_1, \ldots, i_\epsilon$ at the base points $X_1, \ldots, X_\epsilon$, as the components of a vector in a real affine space of $\epsilon + 1$ dimensions; and interpret the grade

$$\nu = m^2 - \sum i^2$$

as the square of the length of the vector, so that the intersection number

$$mm' - \sum ii'$$

of two systems represented by vectors $(m, i_1, \ldots, i_\epsilon)$, $(m', i'_1, \ldots, i'_\epsilon)$ is to be regarded as the scalar product of the two vectors. (I have used this device elsewhere (7) at some length.) This metric is of the same kind as that introduced by Minkowski[5] for the special relativity theory, a system of positive grade corresponding to a "time-like" and one of negative grade to a "space-like" vector. It is obvious that not more than $\epsilon + 1$ vectors can all be mutually perpendicular, and that of any such maximal set of perpendicular vectors just one must be time-like and the rest space-like. The vector $(3,1, \ldots, 1)$ representing the system of cubics through the base points is space-like if $\epsilon > 9$, time-like if $\epsilon < 9$, since the virtual grade of this system is $9 - \epsilon$. But the vectors representing the n fundamental curves of $|\phi|$ are all perpendicular to this latter vector, since, for a curve of virtual grade $\nu = -2$ and genus $\pi = 0$,

$$3m - \sum i = \nu - 2\pi + 2 = 0;$$

and they are perpendicular to each other, since two fundamental curves have no intersection outside of the base points. It follows that $n < \epsilon$, or if $\epsilon > 9$ then $n < \epsilon - 1$, since in this case one of the space-like vectors in a maximal

[5]For this geometry see, e.g., (12).

perpendicular set must be $(3,1,\ldots,1)$, which is not a fundamental curve; if $\epsilon = 9$ there can still be only $\epsilon - 1$ space-like vectors all perpendicular to each other and to $(3,1,\ldots,1)$, as one of the vectors perpendicular to this latter is itself,[6] so that in this case also $n < \epsilon - 1$. Since $\pi' = n - \epsilon + 4$ however, this means that

$$\pi' < 4, \text{ or } \pi' < 3 \text{ if } \epsilon > 9.$$

We have thus to seek a regular system $|\phi|$ satisfying

$$\nu' = 2n - \epsilon + 1, \quad \pi' = n - \epsilon + 4 < 4 \ (\pi' < 3 \text{ if } \epsilon > 9), \quad \rho' = n - 2 > 2,$$

and accordingly

$$\rho' - \pi' - \epsilon + 6 = 0.$$

All regular linear systems of genus < 4 and freedom > 2 are known; those of genus 0, 1 are classical and may be found in many standard works, e.g. (**3**, pp. 280, 320); those of genus 2, 3 were studied by Castelnuovo[7] (**1**), (**2**), and myself (**5**); those of genus 4 by Roth (**13**) and myself (**4**), (**5**). They are listed in various places, but the most convenient references are perhaps (**11**) for $\pi' = 3$ and (**10**) for $\pi' = 4$. In the table below are listed the forms to which all linear

π'	Symbol		ρ'	ϵ	$\rho' - \pi' - \epsilon + 6$	
0	$m\ (m-1, 1^k)$	$0 < k < m-1$	$2m - k$	$k + 1$	$2m + 5 - 2k$	
	1		2	0	8	
	2		5	0	11	
1	$3\ (1^k)$	$0 < k < 7$	$9 - k$	k	$14 - 2k$	A
	$4\ (2^2)$		8	2	11	
2	$4\ (2, 1^k)$	$0 < k < 9$	$11 - k$	$k + 1$	$14 - 2k$	B
	$6\ (2^8, 1^k)$	$0 < k < 1$	$3 - k$	$k + 8$	$-1 - 2k$	
3	$5\ (3, 1^k)$	$0 < k < 12$	$14 - k$	$k + 1$	$16 - 2k$	C
	$4\ (1^k)$	$0 < k < 12$	$14 - k$	k	$17 - 2k$	
	$6\ (2^7, 1^k)$	$0 < k < 4$	$6 - k$	$k + 7$	$2 - 2k$	D
4	$6\ (4, 1^k)$	$0 < k < 7$	$17 - k$	$k + 1$	$18 - 2k$	
	$6\ (3^2)$		15	2	15	
	$5\ (2^2, 1^k)$	$0 < k < 6$	$14 - k$	$k + 2$	$14 - 2k$	
	$6\ (2^6, 1^k)$	$0 < k < 3$	$9 - k$	$k + 6$	$5 - 2k$	
	$9\ (3^8)$		6	8	0	E

[6]I have discussed the peculiarities of the metric in this special case $\epsilon = 9$ at some length in (**7**).

[7]Strictly, he studied the corresponding rational surfaces, but the classification of linear systems originated with these investigations.

systems satisfying the above inequalities can be reduced by Cremona transformation. In the column headed "symbol" the order m is written for clarity outside the parenthesis, and the numbers within are the multiplicities $(i_1, \ldots, i_s)$, except that s consecutive i's within the parentheses are abbreviated as i^s.

It will be seen on inspection of the last column that the relation

$$\rho' - \pi' - \epsilon + 6 = 0$$

is only satisfied by the five systems marked A, B, C, D, E in the margin, for $k = 7, 7, 8, 1$ respectively in the first four cases. From each of these, increasing the order m by 3, and each of the base multiplicities $(i_1, \ldots, i_s)$ by 1, we obtain the symbol for the system $|\phi|$ to which the given system $|\phi'|$ is adjoint; and these we can tabulate as follows:

	Symbol	ν	π	n
A	$6\ (2^7)$	8	3	4
B	$7\ (3, 2^7)$	12	5	6
C	$8\ (4, 2^8)$	16	7	8
D	$9\ (3^7, 2)$	14	6	7
E	$12\ (4^8)$	16	7	8

These systems all have the required relations between their numerical characteristics; but of course they still need to be investigated as to the possibility of choosing the base points in such a way as to make actual the required set of n rational curves of virtual grade -2. In the last two cases this is impossible, as can easily be seen from the following considerations:

Each of the systems D, E has 8 base points, and they require respectively 7 and 8 actual rational curves of grade -2. It is easy of course to find a set of 8 virtual systems of genus 0 and grade -2, every two of which have virtual intersection number 0; but not more than six of these can be made actual by any configuration of the base points. For the system $6(2^8)$ with the same base points has as its projective model a double quadric cone in [3], branching along a sextic curve of genus 4, intersection of the cone with a cubic surface which does not pass through its vertex (the latter, which corresponds to the ninth associated point of the base points, being also an isolated branch point) (**3**, pp. 364-365). Every actual rational curve of grade -2 whose intersection number with every other such curve is 0, corresponds to a conical node on the surface, i.e. to a double point of the branch curve (**6**, p. 457); and this curve can clearly have six double points (by degenerating into three plane sections of the cone) and no more.

Cases A, B, C, on the other hand give the following specializations respectively:

(xviii). The system $6(2^7)$ requires four actual curves of grade -2. This can be achieved by letting six of the base points be the vertices of a complete quadrilateral; the branch curve is 12-ic with quadruple points at all seven base

points, and the canonical pencil consists of the lines through the seventh base point. Alternatively, A_2A_3, A_4A_5 may coincide in [2,2] points, the lines A_2A_3, A_4A_5 meeting in A_1; the branch curve is again 12-ic with quadruple points at all seven base points, and the canonical pencil consists of the conics through A_2, A_4, A_6, A_7. The two mappings are Cremona equivalent. The sextics with seven double base points are well known in general to represent the quadric section of the cone V_3^4 in [6] projecting a Veronese surface from a point; this specialization makes the secant quadric the cone Γ_5^2 with [3] vertex Ω_3, the intersections of the latter with V_3^4 being the four nodes of Φ^8, of which f^{16} is a general quadric section.

(xix) The system $7(3,2^7)$ requires six actual curves of grade -2. This can be achieved by making the double base points A_2A_3, A_4A_5, A_6A_7, coincide in [2,2] points, with the lines A_2A_3, A_4A_5, A_6A_7 all passing through the triple base point A_1. The branch curve is 13-ic with quintuple point at A_1 amd quadruple at all seven double base points; the canonical pencil consists of conics through A_2, A_4, A_6, A_8. On Φ^{12}, the branch curve f^{20} is a quadric section residual to one rational quartic of the pencil represented by lines through A_1.

(xx) The system $8(4,2^8)$ requires eight actual curves of grade -2, which are obtained by making the double base points A_2A_3, A_4A_5, A_6A_7, A_8A_9 coincide in four [2,2] points, the lines A_2A_3, A_4A_5, A_6A_7, A_8A_9 all passing through the quadruple base point A_1. The branch curve is 14-ic with sextuple point at A_1 and quadruple at all eight double base points; the canonical pencil consists of conics through A_2, A_4, A_6, A_8. On Φ^{16} the branch curve f^{24} is a quadric section, residual to two curves of the pencil of rational quartics represented by lines through A_1.

It is interesting to note that all the exceptional cases we have found for $p > 2$, twenty in all, with a solitary exception, fall under a single formula. We observe that the cubics $3[1^6]$ whose base points are the vertices of a complete quadrilateral is Cremona transformable into the system $3[1^6]$, two pairs of whose base points coincide in [1,1] points, the lines joining these both passing through the same fifth base point; and that the system of cubic sections of a quadric cone, with a [1,1] base point, projects from this point into the plane system of quintics with a [2,2] point and a simple base point in the same line. Thus the systems $|\phi|$ we have considered, save that for $p = 6$, $n = 9$, are equivalent to plane systems of curves of order $m + 4$ $(m > 0)$ with

(i) An m-ple base point A;

(ii) $m[2,2]$ points B_{2i-1}, B_{2i} $(i = 1, \ldots, m)$, the lines $B_{2i-1}B_{2i}$ all passing through A;

(iii) h double base points $C_1, \ldots, C_h$;

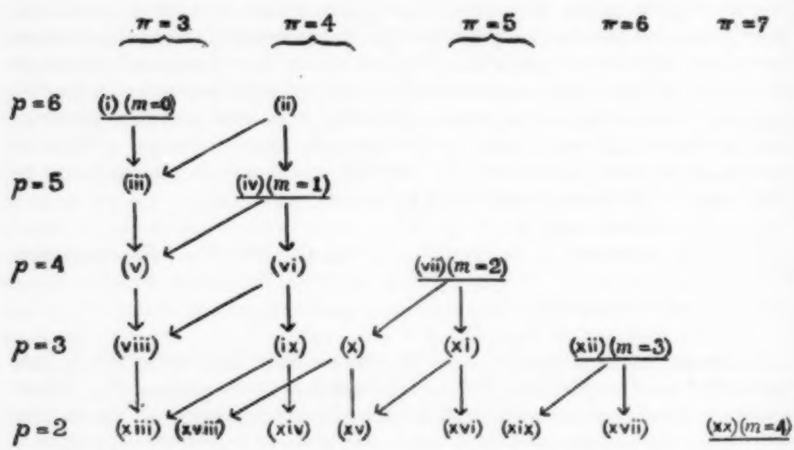
(iv) j [1,1] base points $D_{2i-1}D_{2i}$ $(i = 1, \ldots, j)$.

The branch curve is of order $m + 10$, with an $(m + 2)$-ple point at A, [4,4] points at $B_{2i-1}B_{2i}$, quadruple points at C_i, and [3,3] points at $D_{2i-1}D_{2i}$. There are $2m + j$ nodes on the surface Φ, model of $|\phi|$, represented by the m lines $AB_{2i-1}B_{2i}$, the neighbourhoods of the m points B_{2i-1}, and those of the j points

D_{2t-1}; and these are to be isolated branch points, so that the total branch curve of the double plane is of order $2m + 10$, with $(2m + 2)$-ple point at A, [5,5] points at $B_{2t-1}\,B_{2t}$, quadruple points at C_t, and [3,3] points at $D_{2t-1}\,D_{2t}$. The canonical system is thus of order $m + 2$, with m-ple point at A, double at B_{2t-1}, and simple at B_{2t}, C_t, D_{2t-1}; from this system the m lines $AB_{2t-1}\,B_{2t}$ separate out leaving that of conics with $m + h + j$ simple base points at B_{2t-1}, C_t, D_{2t-1}, of which any two (with the m lines) form a curve of $|\phi|$.

$|\phi|$ has grade $2n$ where $n = 8 - 2h - j$, and genus $\pi = m - n + 3$; and the double Φ^{2n} with the defined branch curve and isolated branch points at the $2m + j$ nodes is a bicanonical surface of genus $p = 6 - m - h - j$ and linear genus $n + 1$. There is on Φ^{2n} a pencil (for $m = 0$ a homoloidal net) of rational quartics represented by the lines through A (which for $m = 0$ is absent), and for $j = 0$ the branch curve f is coresidual to a quadric section together with $2 - m$ of these curves. For $m > 2$, $j > 1$ the $n - 2p + 4 = 2m + j$ nodes fall into two sets; for m curves of the pencil (represented by the neighbourhoods of B_{2t}) consist of a repeated conic joining two nodes, while j curves of it consist of two conics (represented by the neighbourhood of D_{2t} and the line AD_{2t-1}) meeting in a node. For $m = 1$ the nodes are of two kinds again, but the distinction is not quite the same, as the pencil of quartics is not unique; here however the representation by cubic sections of a quadric cone, with $j + 1$ [3,3] points, makes it clear that there is a unique pencil of rational cubics, passing through one of the nodes, and of which $j + 1$ members break up into a conic through this node and a line, meeting in one of the other $k + 1 = n - 2p + 3$ nodes.

Our 20 exceptional cases can now be tabulated as is shown below, values of p reading downwards and those of $\pi = n - p + 1$ across. A vertical arrow indicates the imposition of a [3,3] point on the branch curve and a [1,1] base point



on $|\phi|$, with unit decrease of n and p; an oblique arrow indicates the imposition of a quadruple point on the branch curve and a double base point on $|\phi|$ with unit decrease of p and decrease of 2 in n. The cases corresponding to $m = 0$, 1, 2, 3, 4 respectively, with $h = j = 0$, are underlined and the values of m indicated. We note that the sequences for $m = 1$, $h > 1$ are the same as for $m = 0$, with h diminished by 1 and j increased by 2; and those for $m = 0$, $h > 1$ (and of course $m = 1$, $h > 2$) are included in the standard case. Apart from this the sequences for different values of m are wholly distinct.

We could of course continue this table a line further, obtaining surfaces of genus 1 whose unique canonical curve (when deprived of exceptional constituents) is irreducible and hyperelliptic; and the standard case would give similar surfaces for all values of the linear genus. There is however no reason to suppose that we should obtain a complete list of such surfaces, since so far as I know there is no reason to suppose that the involution on the unique canonical curve would be contained in an involution on the whole surface.

Similar considerations apply to the surfaces of genus zero which would be obtained by continuing the above table, or the specification of the standard case, a stage further again. It is interesting to note however that we can obtain in this way, if not a complete enumeration, at least samples of surfaces of genus zero and arbitrarily high bigenus.

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SOME THEOREMS ON DIFFERENCE SETS

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A set $a_1, \ldots, a_k$ of different residues mod v is called a difference set (v, k, λ) $(v > k > \lambda)$ if the congruence $a_i - a_j \equiv d \pmod{v}$ has exactly λ solutions for $d \not\equiv 0 \pmod{v}$. Singer [**4**] has demonstrated the existence of a difference set $(v, k, 1)$ if $k - 1$ is a prime power, and difference sets for $\lambda > 1$ have been constructed by various authors; but necessary and sufficient conditions for the existence of a (v, k, λ) are not known. It has not been possible so far to find a difference set with $\lambda = 1$ if $k - 1$ is not a prime power and it has therefore been conjectured that no such difference set exists. The condition

$$k(k-1) = \lambda(v-1) \tag{1}$$

is trivial. Owing to the efforts of Hall [**2**] and Hall and Ryser [**3**] efficient necessary conditions are now available by which a large number of (v, k, λ) can be shown to be impossible. Hall [**2**] in particular succeeded in eliminating all doubtful cases of $(v, k, 1)$ with $k - 1 < 100$ and this bound could easily be extended upward. It is the purpose of the present paper to improve some of the results of Hall [**2**] and Hall and Ryser [**3**].

A number t is called a multiplier of (v, k, λ) if $\{ta_i\} \equiv \{a_j + s\} \pmod{v}$ for some s. Hall and Ryser [**3**] generalizing a theorem of Hall [**2**] proved that every prime divisor p of $k - \lambda = n$ is a multiplier provided $p > \lambda$. The restriction $p > \lambda$ can sometimes be obviated by remembering that the residues which are not in (v, k, λ) form a $(v, v - k, v - 2k + \lambda)$ with the same multiplier system as (v, k, λ).

We shall prove the following:

THEOREM 1. *If t is of even order with respect to a prime divisor q of v then n is a square if* $\left(\dfrac{t}{q}\right) = -1$. *If* $\left(\dfrac{t}{q}\right) = +1$ *then $n = b^2$ or a^2q^3, where a, b are integers.* Thus always $n = b^2$ if $n \not\equiv 0 \pmod{q}$.

Proof. Let t have order $2f$ with respect to q then $t^f \equiv -1 \pmod{q}$. We put

$$\theta(x) = x^{a_1} + \ldots + x^{a_k}.$$

Since t is a multiplier, we have for some s,

$$\theta(x^{t^f}) \equiv x^s\theta(x) \qquad \operatorname{mod}(x^v - 1). \tag{2}$$

Substituting a primitive qth root of unity ζ for x we have

$$\theta(\zeta^{t^f}) = \theta(\zeta^{-1}) = \zeta^s\theta(\zeta). \tag{3}$$

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The prime q must be odd, hence $2r \equiv s \pmod{q}$, and since

$$\theta(x)\theta(x^{-1}) \equiv n + \lambda(1 + \ldots + x^{v-1}) \qquad \operatorname{mod}(x^v - 1)$$

it follows that

$$(\zeta^r\theta(\zeta))^2 = n. \tag{4}$$

In the field $\mathfrak{F}(\zeta)$ generated by ζ over the field of rational numbers the field $\mathfrak{F}(\sqrt{\pm q})$ is the only quadratic subfield. Hence either n is a square or $n = a^2q$. In the latter case we have

$$(\zeta^r\theta(\zeta)) = \pm a\sqrt{q}. \tag{4a}$$

The Galois group of $\mathfrak{F}(\zeta)$ over $\mathfrak{F}(\sqrt{q})$ is the group of automorphisms $\zeta \to \zeta^a$ where a is a quadratic residue mod q. If $\left(\frac{t}{q}\right) = -1$ then $\zeta \to \zeta^t$ maps $\sqrt{q}$ into $-\sqrt{q}$. Hence if t is a multiplier,

$$\zeta^{rt}\theta(\zeta^t) = \zeta^{rt+s_1}\theta(\zeta) = \mp a\sqrt{q},$$
$$\zeta^{rt+s_1-r} = -1,$$

but this is impossible since q is odd.

The congruences $n \equiv 0 \pmod{q}$, $v \equiv 0 \pmod{q}$ imply $n \equiv 0 \pmod{q^2}$, since

$$\lambda v = n^2 + (2\lambda - 1)n + \lambda^2; \tag{5}$$

but $n \equiv 0 \pmod{q^2}$ and $n = a^2q$ imply $a \equiv 0 \pmod{q}$, which proves the second part of Theorem 1.

THEOREM 1a. *If under the conditions of Theorem* 1 *we have* $v = q$, *then* $k = v - 1$.

For then $(v, n) = 1$ and following the proof of Theorem 1 we are led to the equation

$$\zeta^r\theta(\zeta) = \pm b, \qquad b \text{ integral.}$$

But this relation is impossible unless $k = v - 1$.

Theorem 1 is a considerable improvement over Hall's Corollary 4.7 and Hall and Ryser's Theorem 3.2.

Theorem 1 has many applications. We give a few indicating its use. In the following corollaries let p always denote a prime divisor of n which exceeds λ and suppose that (v, k, λ) exists. We also assume $v \equiv 1 \pmod{2}$ since for $v \equiv 0 \pmod{2}$, n must always be a square [1].

COROLLARY 1. *If* $\lambda = 1$ *and* $n \equiv n_1$ *or* n_1^2 mod $(n_1^2 + n_1 + 1)$ *and p is of even order with respect to* $n_1^2 + n_1 + 1$, *then n is a square.*

For then $v = n^2 + n + 1 \equiv 0 \bmod (n_1^2 + n_1 + 1)$. Thus p is of even order with respect to a prime divisor q of v. Also in this case $(v, n) = 1$.

For instance n must be a square in the following cases:

$$\begin{array}{llll}
n \equiv 1 & \pmod{3} & p \equiv 2 & \pmod{3} \\
n \equiv 2,\ 4 & \pmod{7} & p \equiv 3, 5, 6 & \pmod{7} \\
n \equiv 3,\ 9 & \pmod{13} & p \equiv 2, 4, 5, 6, 7, 8, 10, 11, 12 & \pmod{13} \\
n \equiv 5,\ 25 & \pmod{31} & \left(\frac{p}{31}\right) = -1 & \\
n \equiv 6,\ 36 & \pmod{43} & \left(\frac{p}{43}\right) = -1 & \\
n \equiv 7,\ 11 & \pmod{19} & \left(\frac{p}{19}\right) = -1 &
\end{array}$$

and so forth.

COROLLARY 2. *If a multiplier is quadratic non-residue modulo a prime divisor of v then n is a square. Moreover, if v is prime then $k = v - 1$.*

COROLLARY 3. *If*

$$\left(\frac{(-1)^{\frac{1}{2}(v-1)}\lambda}{p}\right) = -1$$

then n is a square; if further v is a prime then (v, k, λ) is impossible.

For by (5) we have

$$\left(\frac{\lambda v}{p}\right) = \left(\frac{\lambda^2}{p}\right) = +1;$$

hence

$$\left(\frac{v}{p}\right) = \left(\frac{\lambda}{p}\right).$$

But

$$\left(\frac{p}{v}\right) = (-1)^{\frac{1}{2}(p-1)\frac{1}{2}(v-1)}\left(\frac{v}{p}\right) = \left(\frac{(-1)^{\frac{1}{2}(v-1)}\lambda}{p}\right),$$

and the corollary follows from Theorems 1 and 1a.

The case (91, 45, 22) already eliminated by Hall and Ryser is also quickly disposed of by Theorem 1, since $23 \equiv -3 \pmod{13}$ and -3 has the order 6 (mod 13).

We shall call a prime p an extraneous multiplier if p is a multiplier but $n \not\equiv 0 \pmod{p}$. We shall prove

THEOREM 2. *The prime p is a multiplier if and only if*

$$\theta(x)^p \equiv x^s\theta(x) \qquad \operatorname{modd}(p, x^v - 1). \tag{6}$$

If p is an extraneous multiplier then

$$\theta(x)^{p-1} \equiv x^s \qquad \operatorname{modd}(p, x^v - 1) \tag{6'}$$

if $k \not\equiv 0 \pmod p$, *and*

$$\theta(x)^{p-1} \equiv x^{s} - T(x) \qquad \text{modd}(p, x^v - 1), \tag{6''}$$

$T(x) = 1 + x + \ldots + x^{v-1}$, *if* $k \equiv 0 \pmod p$.

Proof. If p is a multiplier we have

$$x^s\theta(x) \equiv \theta(x^p) \equiv \theta(x)^p \qquad \text{modd}(p, x^v - 1).$$

On the other hand, $\theta(x)^p \equiv x^s\theta(x)$, modd $(p, x^v - 1)$, implies $\theta(x^p) \equiv x^s\theta(x)$, modd $(p, x^v - 1)$. Since $\theta(x^p)$ and $x^s\theta(x)$ are polynomials whose coefficients are either 1 or 0, it follows from this that

$$\theta(x^p) \equiv x^s\theta(x) \qquad \text{mod}(x^v - 1).$$

Hence p is a multiplier.

If p is an extraneous multiplier we multiply (6) by $\theta(x^{-1})$ and obtain

$$\theta(x)^{p-1}(n + \lambda T(x)) \equiv x^s(n + \lambda T(x)) \qquad \text{modd } (p, x^v - 1), \tag{7}$$

$$n\theta(x)^{p-1} + \lambda k^{p-1}T(x) \equiv x^s(n + \lambda T(x)) \qquad \text{modd } (p, x^v - 1). \tag{7'}$$

If $k \not\equiv 0 \pmod p$ then $k^{p-1} \equiv 1 \pmod p$. If $k \equiv 0 \pmod p$ then $n \equiv -\lambda \pmod p$. Also $x^sT(x) \equiv T(x)$, mod $(x^v - 1)$, and the second part of the theorem follows easily from (7) and (7′).

COROLLARY 1. *If 2 is a multiplier for* (v, k, λ) *then either* $n \equiv 0 \pmod 2$ *or* $k = v - 1$.

For otherwise Theorem 2 gives either

$$\theta(x) \equiv x^s \qquad \text{modd } (2, x^v - 1),$$

or

$$\theta(x) \equiv x^s + T(x) \qquad \text{modd } (2, x^v - 1)$$

and the corollary follows.

COROLLARY 2. *If 3 is a multiplier for* $(v, k, 1)$ *then* $n \equiv 0 \pmod 3$.

For otherwise either

$$\theta(x)^2 \equiv x^s \qquad \text{modd } (3, x^v - 1), \tag{8}$$

or

$$\theta(x)^2 \equiv x^s - T(x) \qquad \text{modd } (3, x^v - 1). \tag{8'}$$

But x^m occurs in $\theta(x)^2$ only if $m = a_i + a_j$ and then exactly twice if $i \neq j$ and exactly once if $i = j$, whilst x^m does not occur for exactly $\frac{1}{2}n(n+1)$ values of m. Thus (8) and (8′) are both impossible, and the corollary follows.

The following two theorems serve to show the non-existence of $(v, k, 1)$ in a large number of doubtful cases.

Theorem 3. *If t_1, t_2, t_3, t_4 are multipliers of $(v, k, 1)$ such that $t_1 + t_2 \equiv t_3$, $t_2 \not\equiv t_4 \pmod{v}$ then $t_1 + t_4$ is not a multiplier.*

For in this case we have a difference set $a_1, \ldots, a_k$ which remains fixed under all multipliers [2]. If $t_1 + t_4 \equiv t_5 \pmod{v}$ is a multiplier, then for every a in this difference set

$$at_1 + at_2 \equiv at_3 \equiv a_k \pmod{v},$$

$$at_1 + at_4 \equiv at_5 \equiv a_l \pmod{v},$$

$$a_k - a_l \equiv at_2 - at_4 \pmod{v}.$$

Hence, since $\lambda = 1$, either $at_2 \equiv a_k \pmod{v}$ which implies $a \equiv 0$ or $at_2 \equiv at_4$, $a(t_2 - t_4) \equiv 0 \pmod{v}$. Hence for all a we have $a(t_2 - t_4) \equiv 0 \pmod{v}$; but since every $m \equiv a_i - a_j \pmod{v}$ it follows that $t_2 - t_4 \equiv 0 \pmod{v}$.

Corollary 1. *If 2, p, q are multipliers for $(v, k, 1)$ and $p \not\equiv q \pmod{v}$ then $p + q$ is not a multiplier.*

This follows since $p + p = 2p$ is a multiplier.

Corollary 2. *If 2 and $2^k + 1$ are multipliers then $2^k \equiv 1 \pmod{v}$. If 2 and $2^k - 1$ are multipliers then $2^k - 1 \equiv 1 \pmod{v}$.*

This follows at once from Corollary 1 with $p = 1$.

Theorem 4. *If t_1, t_2, t_3, t_4 are multipliers for $(v, k, 1)$ and $(t_1 - t_2) = (t_3 - t_4)$ then*

$$(t_1 - t_2)(t_1 - t_3) \equiv 0 \pmod{v}. \tag{9}$$

For again let $a_1, \ldots, a_k$ be the set that remains fixed under all multipliers. Then for any a in this set,

$$t_1a - t_2a \equiv t_3a - t_4a \pmod{v}.$$

Hence either $t_1a \equiv t_2a \pmod{v}$ or $t_1a \equiv t_3a \pmod{v}$. Hence for all a, and therefore for every number m, we must have

$$(t_1 - t_2)(t_1 - t_3)m \equiv 0 \pmod{v},$$

whence the theorem.

Theorem 4 was extensively used, but not explicitly stated, by Hall [2].

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ON A GALOIS CONNECTION BETWEEN ALGEBRAS OF LINEAR TRANSFORMATIONS AND LATTICES OF SUBSPACES OF A VECTOR SPACE

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1. Introduction. Representation theory has contributed much to the study of linear associative algebras. The central problem of representation theory *per se* is the determination for each algebra of all its indecomposable representations. This turns out to be a much deeper problem than the classification of algebras, in the sense that there are algebras for which any "internal question" can be answered but for which the number and nature of representations is almost completely unknown, or if known is much more complicated than the internal theory. This can be illustrated by the example of a commutative algebra of order three for which the representation theory can be shown to be essentially the same as the problem of classifying pairs of rectangular matrices under equivalence. (This algebra has indecomposable representations of every integral degree.)

Detailed study (as yet unpublished) of the representations of certain classes of algebras has led me to consider the possibility of searching for connections between representation theory and lattice theory. The present note is devoted to setting up the machinery for certain phases of such an investigation.

Notation and definitions. Let $\mathfrak{k}$ be a sfield and V a right $\mathfrak{k}$-space of dimension n. If $v_1, \ldots, v_n$ are a basis for V then any vector v in V can be written in the form $v = v_1a_1 + \ldots + v_na_n$ where $a_1, \ldots, a_n$ are uniquely determined scalars (i.e. elements of $\mathfrak{k}$) called the coordinates of v relative to the given basis for V. This can be written in the matrix form as $v = \|v_j\| \cdot \|a_i\|$ where $\|v_j\|$ denotes the 1 by n (row) matrix made up of the basis vectors and $\|a_i\|$ denotes the n by 1 (column) matrix made up of the coordinates. (In describing any matrix we shall use the subscript "i" for row index and "j" for column index.) Then for any vector v and scalar a, va is the vector with coordinate matrix $\|b_i\| = \|a_ia\|$.

We denote by $\mathfrak{T}$ the set of all linear transformations $\mathfrak{a}$ (i.e., $\mathfrak{k}$-endomorphism) of V into itself. We write the linear transformations as left operators, and then the commutativity of linear transformations with the scalar multiplications take the form $(\mathfrak{a}v)a = \mathfrak{a}(va)$. To express the linear transformations in matrix form we use the formula

$$\begin{aligned}\mathfrak{a}v &= \mathfrak{a}\,(\|v_j\|\cdot\|a_i\|) = (\mathfrak{a}\|v_j\|)\|a_i\| = \|\mathfrak{a}v_j\|\cdot\|a_i\| \\ &= (\|v_j\|T_{\mathfrak{a}})\|a_i\| = \|v_j\|(T_{\mathfrak{a}}\|a_i\|).\end{aligned}$$

Here $T_{\mathfrak{a}}$ is, of course the matrix whose jth column is the coordinate matrix of

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αv_j. Conversely, the same formula read in reverse shows that every n by n $\mathfrak{k}$-matrix T defines a linear transformation α. We make $\mathfrak{T}$ into a right $\mathfrak{k}$-space of dimension n^2 by the definition $\alpha a = \beta$, where β is the linear transformation for which

$$T_\beta = T_\alpha \left\| \begin{matrix} a & 0 \\ 0 & a \end{matrix} \right\|.$$

The set $\mathfrak{R}$ of all subspaces of V is a complemented modular lattice of (lattice) dimension n. With any subalgebra $\mathfrak{A}$ of $\mathfrak{T}$ we associate the sublattice $\mathfrak{L} = \mathfrak{A}^*$ of $\mathfrak{R}$ consisting of all subspaces of V invariant under $\mathfrak{A}$. With any sublattice $\mathfrak{L}$ of $\mathfrak{R}$ we associate the subalgebra $\mathfrak{A} = \mathfrak{L}^+$ consisting of all linear transformations α in $\mathfrak{T}$ for which each element W of $\mathfrak{L}$ is an invariant subspace. A subalgebra $\mathfrak{A}$ of $\mathfrak{T}$ is said to be *closed* if $(\mathfrak{A}^*)^+ = \mathfrak{A}$. A sublattice $\mathfrak{L}$ of $\mathfrak{R}$ is said to be closed if $(\mathfrak{L}^+)^* = \mathfrak{L}$.

The mappings "*" and "+" constitute a Galois connection [1, p.56] between the subalgebras of $\mathfrak{T}$ and the sublattices of $\mathfrak{R}$ (i.e., both * and + invert inclusion and for all $\mathfrak{A}$ we have $(\mathfrak{A}^*)^+ \supseteq \mathfrak{A}$ and for all $\mathfrak{L}$ we have $(\mathfrak{L}^+)^* \supseteq \mathfrak{L}$). The mappings $\mathfrak{A} \to (\mathfrak{A}^*)^+$ and $\mathfrak{L} \to (\mathfrak{L}^+)^*$ are accordingly closure operations [1, p. 49] in which the closed elements are just the images under * and +, i.e., $\mathfrak{A}$ is closed if and only if there exists an $\mathfrak{L}$ for which $\mathfrak{A} = \mathfrak{L}^+$ and $\mathfrak{L}$ is closed if and only if there exists an $\mathfrak{A}$ for which $\mathfrak{L} = \mathfrak{A}^*$.

The main purpose of this paper is the beginning of the study of the mappings * and +. Among the problems considered (but not completely solved) are the determination of intrinsic characterizations of closure, and the ways in which properties of the subalgebras and sublattices can be traced in their images under * and + respectively.

The main results of the paper are the two necessary conditions (Theorems 1 and 2) that a lattice be closed, given in §§3 and 4; §§5 and 6 deal with the special case of distributive lattices. Every distributive lattice is closed; the closed algebras whose lattices are distributive are characterized and some sufficient conditions are obtained that an algebra $\mathfrak{A}$ should define a distributive lattice $\mathfrak{A}^*$. Section 7 contains some examples and conjectures.

2. Some elementary properties of mappings + and *. Suppose that $\mathfrak{L} = \mathfrak{A}^*$ is complemented. In the language of representation theory this says that V is a completely reducible representation space for $\mathfrak{A}$. Since $\mathfrak{A}$ is defined as an algebra of linear transformations on V we see that V is space for a faithful representation of $\mathfrak{A}$. Now, any algebra which has a faithful completely reducible representation is semi-simple (for radical elements are mapped into zero by each irreducible representation). Conversely, if $\mathfrak{A}$ is semi-simple then V is completely reducible, that is, $\mathfrak{L} = \mathfrak{A}^*$ is complemented. Hence $\mathfrak{L} = \mathfrak{A}^*$ *is complemented if and only if* $\mathfrak{A}$ *is semi-simple.*

If $\mathfrak{A} = \mathfrak{L}^+$ then $\mathfrak{A}$ has a unit element. Since the dimension of a lattice is greater than or equal to the dimension of any sublattice we see that the dimension of $\mathfrak{L}$ is less than or equal to the composition length of V considered as an $\mathfrak{A}$-space. The following example shows that the inequality can occur. Let $\mathfrak{k}$ be the rationals

and let $n = 4$. Let $\mathfrak{L}$ have the elements V, S, T, U, Q, R where S is the set of all vectors with coordinates of the form $(a, b, 0, 0)$; T is the set of all vectors with coordinates of the form $(0, 0, a, b)$; U is the set of all vectors with coordinates of the form (a, b, a, b); Q is the set of all vectors with coordinates of the form $(a, b, 2a, 3b)$; and R is the zero space. Then $\mathfrak{A}$ is the set of all matrices of the form

$$\left\| \begin{matrix} {}^{c}{}_{d} & 0 \\ 0 & {}^{c}{}_{d} \end{matrix} \right\|,$$

and $\mathfrak{A}^*$ has dimension 4.

3. Projective closure. Let A and B be subspaces of V, i.e., elements of the lattice $\mathfrak{N}$. We use the symbols $A \cap B$ and $A \cup B$ respectively to indicate the intersection of A and B and the space spanned by A and B. If B is a subspace of A we say that the pair A, B defines a *quotient*, written A/B. (This should not be confused with the residue class space which we denote by $A - B$.) We shall use quotients only in connection with the concepts of transposed quotients and projective quotients [1, p. 72].

If A/B and C/D are transposes with $A \cup D = C$ and $A \cap D = B$ then any vector v in A is in C and two vectors in A belong to the same coset modulo B if and only if they belong to the same coset of C modulo D. If v is any vector in C there is a vector v' in the same coset modulo D such that v' is also in A. It is easy to see that the mapping $v + B \to v + D$ defined for all vectors v in A is a non-singular linear transformation of the factor space $A - B$ onto $C - D$.

The inverse of this mapping is, of course, also a linear transformation. Hence, with any sequence of transposes leading from an initial quotient A/B to a final quotient C/D we can associate a unique (non-singular) linear transformation of $A - B$ onto $C - D$. We say that such transformations are *lattice induced*. Suppose that in a sublattice $\mathfrak{L}$ of $\mathfrak{N}$ two quotients S/R and T/R are projective with

$$S/R = X_0/Y_0, \quad X_1/Y_1, \quad \ldots, \quad X_k/Y_k = T/R$$

as a sequence of transposes which demonstrate this projectivity [2].

Denote by σ: $s + R \to t + R = \sigma(s + R)$ the mapping thus defined from $S - R$ onto $T - R$ by the above given sequence of X_i/Y_i. Then for each pair $a \in \mathfrak{k}$ and $s \in S$ we define $Q_a(s)$ to be the coset $(s + R) + \sigma(s + R)a$ of $S \cup T$ modulo R. If, now, $q_1 \in Q_a(s_1)$ and $q_2 \in Q_a(s_2)$, then $(q_1b_1 + q_2b_2) \in Q_a(s_1b_1 + s_2b_2)$. Hence, the set Q_a consisting of all vectors lying in any one of the cosets $Q_a(s)$ for some $s \in S$ is a subspace of V, that is, an element of $\mathfrak{N}$. If $a \neq 0$, then Q_a has meet R and join $S \cup T$ with both S and T so that $[R; S, T, Q_a; S \cup T]$ is a projective root [2, p. 147] in $\mathfrak{N}$. Moreover, the mapping $s + R \to Q_a(s)$ is a linear transformation of $S - R$ onto Q_a. We say that Q_a is *projectively related* to $\mathfrak{L}$.

Definition. We say that $\mathfrak{L}$ is *projectively closed in* $\mathfrak{N}$ if $\mathfrak{L}$ contains every space Q_a projectively related to it.

The above process for defining new spaces Q_a can be generalized in the following manner. Let $W \supset S \supset R$ be a chain in $\mathfrak{R}$, let σ be a linear transformation of $S - R$ into $W - R$, and let $a \in \mathfrak{k}$. Then we define Q_a to be the set of all vectors lying in any one of the cosets $Q_a(s) = (s + R) + \sigma(s + R)a$ for $s \in S$. (Of course, spaces thus obtained need not be projectively related to $\mathfrak{L}$ even if R, S, and W all belong to $\mathfrak{L}$.)

LEMMA 1. *Let $\mathfrak{A}$ be any subalgebra of $\mathfrak{T}$, let $W \supset S \supset R$ be a chain in $\mathfrak{L} = \mathfrak{A}^*$, let σ be an operator homomorphism ($\mathfrak{A}$-homomorphism) of $S - R$ into $W - R$, and let $a \in \mathfrak{k}$. Then $Q_a \in \mathfrak{L}$.*

Proof. Let $\mathfrak{a} \in \mathfrak{A}$. Then we have

$$\mathfrak{a}Q_a(s) = \mathfrak{a}[(s + R) + \sigma(s + R)a] = \mathfrak{a}(s + R) + \mathfrak{a}[\sigma(s + R)a]$$

$$= \mathfrak{a}(s + R) + \sigma(\mathfrak{a}s + R)a = Q_a(\mathfrak{a}s),$$

and hence $\mathfrak{a}Q_a \subseteq Q_a$.

The following theorem which is an immediate consequence of this lemma illustrates the importance of the concept of projective closure.

THEOREM 1. *Every closed lattice $\mathfrak{L} = \mathfrak{A}^*$ is projectively closed.*

Proof. If $\mathfrak{L} = \mathfrak{A}^*$ then every lattice induced linear transformation is an operator ismorphism. Suppose that S/R is projective to T/R in $\mathfrak{L}$ and σ is any lattice induced ismorphism of $S - R$ onto $T - R$. Then apply Lemma 1 with $W = S \cup T$ and we see that $Q_a \in \mathfrak{L}$.

THEOREM 2. *Suppose that $\mathfrak{k}$ is an algebraically closed field, and that $\mathfrak{L}$ is projectively closed in $\mathfrak{R}$. Let $\mathfrak{P} = [R; S, T, U; W]$ be a prime projective root in $\mathfrak{L}$, and let σ be the linear transformation of $S - R$ onto $T - R$ induced by any projectivity (in $\mathfrak{L}$) of S/R and T/R. Then, there exists $a \in \mathfrak{k}$ such that $U = Q_a$. Moreover, if τ is any second linear transformation of $S - R$ onto $T - R$ induced by a projectivity (in L) of S/L and T/R then τ is a scalar multiple of σ.*

Proof. Let $u + R$ be any coset of U modulo R. Since $W - R$ is the direct sum of $S - R$ and $T - R$ there exist unique cosets $s + R$ of S modulo R and $t + R$ of T modulo R such that $u + R = (s + R) + (t + R)$. Since $S \cap T = U \cap S = U \cap T = R$ we see that no one of the vectors u, s, or t can belong to R unless all three do. Moreover, since $S \cup T = U \cup S = U \cup T = W$ we see that every coset of S modulo R and similarly every coset of T modulo R must appear exactly once as $u + R$ runs through all of the cosets of U modulo R. If we denote by $\rho(s + R)$ the (unique) coset $(t + R)$ which is paired with $(s + R)$ in the expression for some $(u + R)$ it is clear that ρ is a non-singular linear transformation of $S - R$ onto $T - R$.

Consider the product $\lambda = \sigma^{-1}\rho$; clearly λ is a non-singular linear transformation of $S - R$ onto itself. Since $\mathfrak{k}$ is algebraically closed, λ has at least one eigenvalue a, and since λ is non-singular $a \neq 0$. Let $(s_0 + R)$ be a non-zero

eigenvector of λ, that is, s_0 does not belong to R and $\lambda(s_0+R) = (s_0a+R)$. Then

$$Q_a(s_0) = (s_0+R) + \sigma(s_0a+R) = (s_0+R) + \rho\lambda^{-1}(s_0a+R)$$
$$= (s_0+R) + \rho(s_0+R) \subset U.$$

The assumption of projective closure requires that $Q_a \in \mathfrak{L}$. Now, $U \supseteq U \cap Q_a \supset R$. But since Q_a and U are both prime over R this requires $U = Q_a$, which establishes the first part of the theorem.

To establish the remaining contention it is clearly sufficient to show that σ is a scalar multiple of ρ. For then the same would be true of τ. To show this we observe that for every $s \in S$ we have $Q_a(s) = (s+R) + \sigma(sa+R)$ as the coset of U modulo R in the form of a sum of a coset of S modulo R and a coset of T modulo R. As we have seen above, such an expression is unique, and hence $\sigma(sa+R) = \rho(s+R)$ for every $s \in S$. From this we conclude that $\sigma a = \rho$, as required.

4. A relative imbedding problem. Consider an l dimensional sublattice $\mathfrak{L}$ of an n-dimensional complemented modular lattice $\mathfrak{N}$. If there exists an l-dimensional complemented sublattice $\mathfrak{M}$ of $\mathfrak{N}$ which contains $\mathfrak{L}$ and which has the same projective structure constants [**2**, §2] as $\mathfrak{L}$ we say that $\mathfrak{L}$ has the *relative imbedding property*. In § 7 below we shall give an example to show that not every $\mathfrak{L}$ has this property.

An algebra $\mathfrak{A}$ is said to be cleft [**3**, p. 499] if its radical $\mathfrak{R}$ has a complement $\mathfrak{B}$ in the lattice of all subalgebras of $\mathfrak{A}$; $\mathfrak{B}$ is then necessarily semi-simple.

THEOREM 3. *Let $\mathfrak{k}$ be a field and let $\mathfrak{A}$ be a cleft subalgebra of $\mathfrak{T}$ with unity element. Then $\mathfrak{L} = \mathfrak{A}^*$ has the relative imbedding property.*

Proof. Let $\mathfrak{R}$ be the radical of $\mathfrak{A}$, let $\mathfrak{B}$ be a semi-simple subalgebra of $\mathfrak{A}$ for which $\mathfrak{A} = \mathfrak{R} + \mathfrak{B}$, and let $V = V_l \supset V_{l-1} \supset \ldots \supset V_0 = 0$, be an $\mathfrak{A}$-composition series for V. We may choose a basis for V adapted to this series for which elements of $\mathfrak{B}$ are represented by matrices with zeros in all non-diagonal blocks, such that equivalent irreducible constituents of $\mathfrak{A}$ are in identical form and such that the elements of $\mathfrak{R}$ are represented by matrices with zeros in all blocks below the main diagonal. If the number of distinct irreducible constituents of $\mathfrak{B}$ is equal to the number r of projective classes of prime quotients in $\mathfrak{L}$ then $\mathfrak{M} = \mathfrak{B}^*$ is complemented and will have the same projective structure constants as $\mathfrak{L}$. Moreover, because of the antitone properties of the mapping * we have $\mathfrak{L} \subseteq \mathfrak{M}$.

We now show that if there are less than r distinct irreducible constituents of $\mathfrak{B}$ then we can replace $\mathfrak{A}$ by a larger cleft algebra $\mathfrak{A}'$ whose semi-simple subalgebra B' has exactly r distinct irreducible constituents and such that $\mathfrak{L} = \mathfrak{A}'^*$. Then $\mathfrak{M}' = \mathfrak{B}'^*$ will serve as the imbedding lattice for $\mathfrak{L}$.

Let $\mathfrak{F}$ be one of the irreducible constituents of $\mathfrak{A}$ and let ϵ be the element of $\mathfrak{B}$ which is represented by the identity matrix in $\mathfrak{F}$ and by zero in all irreducible constituents of $\mathfrak{A}$ which are not equivalent to $\mathfrak{F}$. Suppose that $\mathfrak{J} = \{j_1, \ldots, j_s\}$

is the set of all indices j for which the factor spaces $V_j - V_{j-1}$ have ϵ as identity operator and suppose that not all of the quotients V_j/V_{j-1} for j in $\mathfrak{J}$ are projective in $\mathfrak{L} = \mathfrak{A}^*$. Partition the set $\mathfrak{J}$ into two non-empty subsets $\mathfrak{J}_1$ and $\mathfrak{J}_2$ in such a way that indices of projective quotients lie in the same subset. Then for $i = 1, 2$ let ϵ_i be the (unique) element of $\epsilon\mathfrak{T}\epsilon$ which induces the identity mapping on the factor spaces $V_j - V_{j-1}$ for all j in $\mathfrak{J}_i$, and which induces the zero mapping on all factor spaces $V_j - V_{j-1}$ for all j not in $\mathfrak{J}_i$.

LEMMA 2. *The mappings ϵ_1 and ϵ_2 belong to $\mathfrak{A}^{*+}$.*

Proof. Clearly ϵ_1 and ϵ_2 are orthogonal indempotents whose sum is ϵ, and hence either both or neither belong to $\mathfrak{A}^{*+}$. Suppose that neither belongs to $\mathfrak{A}^{*+}$. Then there must be an $\mathfrak{A}$-space U of lowest $\mathfrak{k}$-dimension for which $\epsilon_1 U \not\subseteq U$. This space U must be join-irreducible; let U' be its unique maximal $\mathfrak{A}$-subspace.

Let j be the smallest index for which $U \subseteq V_j$. Then since V_j covers V_{j-1}, U covers $V_{j-1} \cap U$, and consequently $V_{j-1} \cap U = U'$. This shows that V_j/V_{j-1} is a transpose of U/U'.

If j does not lie in $\mathfrak{J}$ we have $\epsilon V_j = \epsilon V_{j-1}$ and hence $\epsilon U = \epsilon U' \subseteq U'$. Then since $\epsilon_1 = \epsilon_1\epsilon$ we have $\epsilon_1 U = \epsilon_1\epsilon U \subseteq \epsilon_1 U'$. Now since (dimension U') < (dimension U) we have $\epsilon_1 U' \subseteq U'$ and hence $\epsilon_1 U \subseteq U' \subset U$, contrary to our hypothesis on U. Hence j lies in $\mathfrak{J}$.

Now, since j lies in $\mathfrak{J}$, $\epsilon V_j \not\subseteq \epsilon V_{j-1}$. We may suppose the notation so chosen that $j \in \mathfrak{J}_1$. Then for $v \in V_j$, we have $\epsilon_1 v \in V_{j-1}$ if and only if $v \in V_{j-1}$, and similarly for $u \in U$ we have $\epsilon_1 u \in U'$ if and only if $u \in U'$.

Let u be a vector of U for which $\epsilon_1 u \notin U$. Since ϵ_1 and ϵ induce the identity mapping on $V_j - V_{j-1}$ there exist vectors $u_1 \in V_{j-1}$ such that $\epsilon_1(u - u_1) = u - u_1$. (For example $u_1 = u - \epsilon_1 u$ has this property.) Let k be the smallest index for which there exists a pair of vectors $u \in U$ and $u_1 \in V_k$ such that $\epsilon_1 u \notin U$ and $\epsilon_1(u - u_1) = u - u_1$. Clearly $k < j$. If u, u_1 is such a pair so is ϵu, ϵu_1, hence we may suppose that $u = \epsilon u$ and $u_1 = \epsilon u_1$. Since $u_1 (= \epsilon u_1)$ does not lie in V_{k-1} we see that $k \in \mathfrak{J}$. If $k \in \mathfrak{J}_1$ then

$$u_2 = u_1 - \epsilon_1 u_1 \in V_{k-1}.$$

Moreover,

$$\begin{aligned}\epsilon_1(u - u_2) &= \epsilon_1 u - \epsilon_1 u_1 + \epsilon_1\epsilon_1 u_1 = \epsilon_1(u - u_1) + \epsilon_1 u_1 \\ &= u - u_1 + \epsilon_1 u_1 = u - u_2,\end{aligned}$$

contrary to the hypothesis that k is minimal. Hence $k \in \mathfrak{J}_2$.

Set $R = U' \cup V_{k-1}$, $S = U \cup V_{k-1}$, $T = U' \cup V_k$, and $W = U \cup V_k$. We contend that these four $\mathfrak{A}$-spaces are distinct and that $R = S \cap T$, $W = S \cup T$. It is obvious that $S \cup T = W$. Now,

$$\begin{aligned}S \cap T &= (U \cup V_{k-1}) \cap (U' \cup V_k) = [U \cap (U' \cup V_k)] \cup V_{k-1} \\ &= [(U \cap V_k) \cup U'] \cup V_{k-1} = U' \cup V_{k-1} = R.\end{aligned}$$

(The simplifications used follow from two applications of the modular law and the fact that, since $k < j$, $U \cap V_k \subseteq U'$.)

We cannot have $W = T$ lest the above chosen vector u lie in $U' \cup V_k \subseteq V_{j-1}$ from which it would follow that $u \in U \cap V_{j-1} = U'$ which contradicts the condition $\epsilon_1 u \notin U$. In order to show that all four spaces are distinct it is now sufficient to show that $W \neq S$. Suppose that $W = S$. Then $u_1 \in S$ and so can be written in the form $u_1 = u' + u_2$ where $u' \in U'$, $u_2 \in V_{k-1}$. Now $u' \in U'$, hence $\epsilon_1(u - u') \notin U$. Moreover,

$$\epsilon_1[(u - u') - u_2] = \epsilon_1(u - u_1) = u - u_1 = (u - u') - u_2.$$

Thus the pair of vectors $(u - u')$, u_2 contradict the minimality of k. This contradiction arises from the assumption $W = S$; hence we conclude that $W \neq S$.

Since j and k both lie in $\mathfrak{J}$ we see that $S - R$ is $\mathfrak{A}$-isomorphic to $T - R$ under some mapping σ. Now, by Lemma 1, $\mathfrak{L}$ contains the space $U = Q_1$, and the projective root $[R; S, T, U; W]$ in $\mathfrak{L}$ can be used to show the projectivity of the quotients S/R and T/R and thus of V_j/V_{j-1} and V_k/V_{k-1}. But this contradicts the construction of $\mathfrak{J}_1$ and $\mathfrak{J}_2$. This contradiction arises from the assumption that the $\mathfrak{A}$-space U is not invariant under ϵ_1 and ϵ_2; and thus completes the proof of the lemma.

LEMMA 3. *The algebra $\mathfrak{A}_1$ generated by $\mathfrak{A}$ and ϵ_1, ϵ_2 is cleft with semi-simple subalgebra $\mathfrak{B}_1$ generated by $\mathfrak{B}$ and ϵ_1, ϵ_2.*

Proof. It is clear from the matrix form of $\mathfrak{B}$ that ϵ_1 and ϵ_2 commute with all elements of $\mathfrak{B}$ as well as with each other, and hence that ϵ_1 and ϵ_2 belong to the centre of $\mathfrak{B}_1$. One consequence of this is that $\epsilon_1\mathfrak{B}$ is a two-sided ideal of $\mathfrak{B}_1$. But from the matrix form of $\mathfrak{B}$ it is clear that $\epsilon_1\mathfrak{B}$ is a simple algebra isomorphic to $\epsilon\mathfrak{B}$. Similarly $(1 - \epsilon_1)\mathfrak{B}$ is also a two-sided ideal of $\mathfrak{B}_1$, and reference to the matrix form of $\mathfrak{B}$ shows that $(1 - \epsilon_1)\mathfrak{B}$ is isomorphic to $\mathfrak{B}$ under the mapping $\beta \rightarrow (1 - \epsilon_1)\beta$. Since ϵ_1 is idempotent and lies in the centre of $\mathfrak{B}_1$ the sum $\mathfrak{B}' = (1 - \epsilon_1)\mathfrak{B} + \epsilon_1\mathfrak{B}$ is direct. Clearly $\mathfrak{B}' \supseteq \mathfrak{B}$, and the equalities $(1 - \epsilon_1)\epsilon = \epsilon_2$ and $\epsilon\epsilon_1 = \epsilon_1$ show that $\mathfrak{B}'$ also contains ϵ_1 and ϵ_2. Hence, $\mathfrak{B}_1 = \mathfrak{B}'$ is semi-simple. We have proved incidentally that $\mathfrak{B}_1$ *has exactly one more simple two-sided ideal than* $\mathfrak{B}$.

If we can now find a nilpotent ideal $\mathfrak{R}_1$ in $\mathfrak{A}_1$ for which $\mathfrak{A}_1$ is the direct sum of $\mathfrak{B}_1$ and $\mathfrak{R}_1$ we will have completed the proof of the lemma. The radical $\mathfrak{R}$ of $\mathfrak{A}$ consists of those elements represented by zeros in all of the irreducible constituents, i.e., of those elements whose matrices are zero in all blocks on or below the main diagonal. Since V has composition length l it follows that $\mathfrak{R}^l = 0$. Moreover, the matrix for each element of $\mathfrak{R}\mathfrak{B}_1$ has zeros in all blocks on or below the main diagonal so that $(\mathfrak{R}\mathfrak{B}_1)^l = 0$. Now, consider the subset

$$\mathfrak{R}_1 = \mathfrak{B}_1(\mathfrak{R}\mathfrak{B}_1) + \mathfrak{B}_1(\mathfrak{R}\mathfrak{B}_1)^2 + \ldots + \mathfrak{B}_1(\mathfrak{R}\mathfrak{B}_1)^l.$$

of $\mathfrak{A}_1$. We remark first that since $1 \in \mathfrak{B}_1$, $\mathfrak{R} \subseteq \mathfrak{R}_1$ and so $\mathfrak{B}_1$ and $\mathfrak{R}_1$ generate $\mathfrak{A}_1$.

Moreover, since $\mathfrak{B}_1^2 = \mathfrak{B}_1$ we have $\mathfrak{B}_1\mathfrak{R}_1 = \mathfrak{R}_1\mathfrak{B}_1 = \mathfrak{R}_1$. An easy induction shows that

$$\mathfrak{R}_1^{\,t} = \mathfrak{B}_1(\mathfrak{R}\mathfrak{B}_1)^t + \mathfrak{B}_1(\mathfrak{R}\mathfrak{B}_1)^{t+1} + \ldots + \mathfrak{B}_1(\mathfrak{R}\mathfrak{B}_1)^l.$$

This shows both that $\mathfrak{R}_1$ is nilpotent and that it is closed under multiplication. To check closure under addition it suffices to recall that the definition of the product $\mathfrak{C}\mathfrak{D}$ of two subsets $\mathfrak{C}$ and $\mathfrak{D}$ is the set of all sums $c_1d_1 + \ldots + c_sd_s$ for the c_i in $\mathfrak{C}$ and the d_i in $\mathfrak{D}$. Putting together all of the above facts we see that $\mathfrak{R}_1$ is a nilpotent two-sided ideal of $\mathfrak{A}_1$. It remains only to show that the sum $\mathfrak{B}_1 + \mathfrak{R}_1$ is direct and equal to $\mathfrak{A}_1$. The intersection of $\mathfrak{B}_1$ and $\mathfrak{R}_1$ is clearly an ideal in $\mathfrak{B}_1$ and is as part of $\mathfrak{R}_1$ either nilpotent or zero. Since $\mathfrak{B}_1$ is semi-simple this intersection must be zero, so the sum is direct. Now $\mathfrak{A}' = \mathfrak{B}_1 + \mathfrak{R}_1$ is clearly a subalgebra of $\mathfrak{A}_1$. Since $\mathfrak{A}_1$ is generated by $\mathfrak{B}_1$ and $\mathfrak{R}_1$ it follows that $\mathfrak{A}_1 = \mathfrak{A}'$ is cleft with semi-simple component $\mathfrak{B}_1$.

Returning now to the proof of Theorem 3, we observe it is a consequence of Lemma 2 that $(\mathfrak{A}_1)^* = \mathfrak{A}^*$. Moreover, we saw in the proof of Lemma 3 that $\mathfrak{A}_1$ has one more class of irreducible representations than $\mathfrak{A}$. Hence, by repeated applications of our construction we will arrive at a cleft algebra $\mathfrak{A}_s$ having exactly as many classes of irreducible representations as $\mathfrak{L}$ has classes of projective prime quotients and for which $(\mathfrak{A}_s)^* = \mathfrak{A}^*$. This completes the proof.

5. The distributive case. We shall show that every distributive lattice is closed and that a closed algebra corresponds to a distributive lattice if and only if its irreducible constituents are total matrix algebras over $\mathfrak{k}$, no two of which are equivalent. The question as to which non-closed algebras correspond to distributive lattices is not settled although some results are given for the case in which $\mathfrak{k}$ is an algebraically closed field.

We review some of the important properties of a distributive lattice [1, Chap. IX]. Let $\mathfrak{L}$ be a distributive sublattice of $\mathfrak{N}$ and let $U_1, U_2, \ldots, U_l$ be the join-irreducible elements of $\mathfrak{L}$ (here $l = \dim \mathfrak{L}$). Let U_j' be the unique element covered by U_j and let $n_j = \dim\,(U_j - U_j')$ $(j = 1, 2, \ldots, l)$. Suppose that the U's have been ordered so that $U_i \subset U_j$ can hold only if $i < j$, and set

$$V_j = U_1 \cup U_2 \cup \ldots \cup U_j \qquad (j = 1, \ldots, l).$$

Then $0 = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_l = V$ is a maximal chain in $\mathfrak{L}$. We choose a basis for V adapted to this chain, and, moreover, such that the jth set of basis elements (i.e., the n_j new ones chosen for V_j in addition to the ones already chosen for V_{j-1}) all lie in U_j. We regard $\mathfrak{A} = \mathfrak{L}^+$ as a matrix algebra in terms of the given basis. Let $A = \|A_{ij}\|$ be an n by n $\mathfrak{k}$-matrix partitioned so that A_{ij} is an n_i by n_j $\mathfrak{k}$-matrix.

LEMMA 4. (i) *If A belongs to $\mathfrak{A} = \mathfrak{L}^+$ then so does each matrix B which can be obtained from A by replacing any set of its submatrices A_{ij} by the zero matrix.* (ii) *If $U_i \subseteq U_j$, and A' is any n_i by n_j $\mathfrak{k}$-matrix then the matrix A having $A_{ij} = A'$ and $A_{hk} = 0$ for $h, k \neq i, j$ belongs to $\mathfrak{A}$.* (iii) *If $U_i \not\subset U_j$ then $A_{ij} = 0$ for all A in $\mathfrak{A}$.*

Proof. The lattice dimension of any element W of $\mathfrak{L}$ is the number of join-irreducible elements of $\mathfrak{L}$ which it contains [1, p. 139]. Suppose that

$$U_{h_1}, \dots, U_{h_t}$$

are the join-irreducible elements contained in W. Then the h_1th, ..., h_tth sets of basis vectors for V taken altogether will be a basis for W. Now let A be as in (ii) and we see that if $U_j \not\subseteq W$ then $AW = 0$ and that if $U_j \subseteq W$ then $AW \subseteq U_i \subseteq U_j \subseteq W$; hence in all cases W is invariant under A, which establishes part (ii) of the lemma. As a consequence of (ii) we see that for each j the idempotent matrix E_j which has i, j component identity and all other components zero is an element of $\mathfrak{A}$. Now (i) follows by consideration of sums of elements E_iAE_j. Finally, to establish (iii) we suppose that A belongs to $\mathfrak{A}$ and observe that for each vector u in U_j the vector $v' = E_iAE_jv$ must again lie in U_j. As an element of the image space of E_i the vector v' must be a linear combination of basis vectors belonging to the ith set. Now, if $A_{ij} \neq 0$ there exist vectors v in U_j for which $v' \neq 0$. Hence, $0 \subset U_i \cap U_j \not\subseteq U_i'$. Since U_i is join-irreducible this requires $U_i \subseteq U_j$. Therefore, we conclude that if $U_i \not\subseteq U_j$ then $A_{ij} = 0$ for all A in $\mathfrak{A}$.

THEOREM 4. *Every distributive lattice is closed. An algebra $\mathfrak{A}$ of linear transformations is closed with $\mathfrak{L} = \mathfrak{A}^*$ distributive if and only if the irreducible constituents of $\mathfrak{A}$ are inequivalent total matrix algebras.*

Proof. Let $\mathfrak{A}$ be an algebra of linear transformations whose irreducible constituents are inequivalent total matrix algebras over a sfield $\mathfrak{k}$. Let

$$0 = V_0 \subset V_1 \subset \dots \subset V_l = V$$

be a composition series for the space V of $\mathfrak{A}$. Then we can select a basis for V adapted to this composition series so that $\mathfrak{A}$ takes the form $\mathfrak{A} = \|\mathfrak{A}_{ij}\|$ where (i) each $\mathfrak{A}_{ij}$ for $i > j$ is zero; (ii) each $\mathfrak{A}_{ii}$ is a total matrix algebra and no two of the $\mathfrak{A}_{jj}$ are equivalent; (iii) each $\mathfrak{A}_{ij}$ for $i < j$ is either zero or the set of all n_i by n_j $\mathfrak{k}$-matrices where n_j = dimension $V_j - V_{j-1}$ = degree $\mathfrak{A}_{jj}$; and (iv) the non-zero $\mathfrak{A}_{ij}$ are completely independent (i.e., they satisfy condition (i) of Lemma 4).

That $\mathfrak{L} = \mathfrak{A}^*$ is distributive of dimension l follows from the fact that the l irreducible constituents of $\mathfrak{A}$ are inequivalent and hence that $\mathfrak{L}$ can contain no projective root. We now search for the join-irreducible elements of $\mathfrak{L}$.

We subdivide the basis vectors for V in terms of which $\mathfrak{A}$ is written into l sets with n_j in the jth set and in such a manner that the elements of the first j sets form a basis for V_j ($j = 1, \dots, l$). Let $\mathfrak{J}_j = \{i_1, \dots, i_t(= j)\}$ be the set of all indices i for which $A_{ij} \neq 0$. Then for each j let U_j be the subspace of V generated by all of the elements in the i_1th, ..., i_tth sets of basis vectors. We now show that $U_1, \dots, U_l$ are the join-irreducible elements of L, and moreover, that $\mathfrak{L}$ is isomorphic to the lattice $\mathfrak{P}$ of subsets of $\mathfrak{J} = \{1, \dots, l\}$ generated by the $\mathfrak{J}_j$.

Suppose that $A_{hk} \neq 0$, let B' be any n_h by n_k $\mathfrak{k}$-matrix, and let B be the element of $\mathfrak{A}$ with $B_{hk} = B'$ and all other components zero. Since by (iv) any A in $\mathfrak{A}$ is a sum of such B, to establish invariance of any space under $\mathfrak{A}$ it is sufficient to test invariance only under all matrices of type B in $\mathfrak{A}$. Now, unless $k \in \mathfrak{J}_j$ we have $BU_j = 0$, and if $k \in \mathfrak{J}_j$ then $BU_j \subseteq U_h$, where of course $h \in \mathfrak{J}_k$. If $\mathfrak{J}_h \subseteq \mathfrak{J}_j$ we would have $U_h \subseteq U_j$ and hence U_j invariant under B. Hence, we conclude that a sufficient condition for invariance of all the U_j is that for each j, $k \in \mathfrak{J}_j$ only if $\mathfrak{J}_k \subseteq \mathfrak{J}_j$. To see that this is true we suppose $h \in \mathfrak{J}_k$, $k \in \mathfrak{J}_j$ and select B as above. Then select an n_k by n_j $\mathfrak{k}$-matrix C' such that $D' = B'C' \neq 0$ and let C be the element of $\mathfrak{A}$ having $C_{kj} = C'$ and all other components zero. Now $D = BC$ has $D_{hj} = D'$ and so $A_{hj} \neq 0$, that is, $h \in \mathfrak{J}_j$. Thus the U_j all belong to $\mathfrak{L}$.

The proof of the join-irreducibility of the U_j rests again on (iv). Let W be any $\mathfrak{A}$-subspace of U_j which is not contained in V_{j-1}. Let w be a vector of W which does not lie in V_{j-1}, let $i \in \mathfrak{J}_j$, and let z be any element of the ith set of basis vectors. Then we can find an n_i by n_j matrix B' such that $Bw = z$ where B is the matrix of $\mathfrak{A}$ with $B_{ij} = B'$ and all other components zero. Hence $W = U_j$. Now in any expression of U_j as a sum of $\mathfrak{A}$-spaces at least one of the summands must contain vectors of U_j which do not lie in V_{j-1} and hence one of the summands is U_j itself. This shows that U_j is join-irreducible.

Let W be any element of $\mathfrak{L}$, let

$$U_{j_1}, \ldots, U_{j_s}$$

be the join-irreducible elements of $\mathfrak{L}$ contained in W, and let $\mathfrak{J}(W) = \{j_1, \ldots, j_s\}$ Then

$$W = U_1 \cup \ldots \cup U_s, \; \mathfrak{J}(W) = \mathfrak{J}_{j_1} \cup \ldots \cup \mathfrak{J}_{j_s}$$

and W has a basis consisting of the elements of the j_1th, $\ldots$, $j_s th$ sets of basis vectors. Moreover, W is uniquely determined by the element $\mathfrak{J}(W)$ of $\mathfrak{P}$. Furthermore, $W_1 \subseteq W_2$ if and only if $\mathfrak{J}(W_1) \subseteq \mathfrak{J}(W_2)$. In other words, the mapping $\sum: W \rightarrow \mathfrak{J}(W)$ of $\mathfrak{L}$ into $\mathfrak{P}$ is 1-1 and isotone. Now, since $\mathfrak{L}$ is a lattice, $\mathfrak{P}' = \sum \mathfrak{L}$ is also a lattice and is isomorphic [**1**, p. 21] to $\mathfrak{L}$. But, since $\mathfrak{L}'$ contains each $\mathfrak{J}_j = \mathfrak{J}(U_j)$, and since $\mathfrak{L}$ is generated by the $\mathfrak{J}_j$, we conclude that $\sum$ is an isomorphism of $\mathfrak{L}$ onto $\mathfrak{P}$.

Now consider $\mathfrak{B} = \mathfrak{A}^{*+} = \mathfrak{L}^+$. According to Lemma 3 we see that $\mathfrak{B}_{ij} = 0$ if and only if $U_i \not\subseteq U_j$, and hence if and only if $\mathfrak{J}_i \not\subseteq \mathfrak{J}_j$ and hence if and only if $\mathfrak{A}_{ij} = 0$. Now, since the non-zero components $\mathfrak{A}_{ij}$ of $\mathfrak{A}$ are completely independent, and since $\mathfrak{A} \subseteq \mathfrak{B}$, we conclude that $\mathfrak{A} = \mathfrak{B}$, that is, $\mathfrak{A}$ is closed.

Finally, to see that every distributive lattice $\mathfrak{L}$ is closed we apply Lemma 4 and note that the lattice $\mathfrak{P}$ defined by $\mathfrak{A} = \mathfrak{L}^+$ is isomorphic to $\mathfrak{L}$ as well as to $\mathfrak{L}^{+*}$; or, even more simply, observe that the join-irreducible subspaces of $\mathfrak{L}$ are again join-irreducible in $\mathfrak{L}^{+*}$.

6. The distributive case for non-closed algebras. We next consider the question "For what algebras $\mathfrak{A}$ is $\mathfrak{L} = \mathfrak{A}^*$ distributive?" approached from the

following point of view. Suppose a distributive sublattice $\mathfrak{L}$ of $\mathfrak{N}$ is given. Then we ask "For which subalgebras $\mathfrak{A}$ of $\mathfrak{B} = \mathfrak{L}^+$ is $\mathfrak{A}^* = \mathfrak{L}$?"

If the irreducible constituents $\mathfrak{A}_{jj}$ of $\mathfrak{A}$ are the same as those of $\mathfrak{B}$ then either $\mathfrak{A} = \mathfrak{B}$ or some component $\mathfrak{A}_{ij}$ of $\mathfrak{A}$ is zero whereas $\mathfrak{B}_{ij} \neq 0$. But then, by Theorem 4, $\mathfrak{A}^* \supset \mathfrak{L}$. Hence, the zero components of $\mathfrak{A}$ must be the same as those of $\mathfrak{B}$. Thus in order to have $\mathfrak{A}^* = \mathfrak{L}$ and $\mathfrak{A} \subset \mathfrak{B}$ we must have equivalence between some of the irreducible constituents of $\mathfrak{A}$. We assume (with no loss of generality) that two irreducible constituents of $\mathfrak{A}$ are equal if they are equivalent.

THEOREM 5. *Let $\mathfrak{L}$ be a distributive sublattice of $\mathfrak{N}$. Let the set $\mathfrak{J} = \{1, \ldots, l\}$ be partitioned into subsets $\mathfrak{J}_1, \ldots, \mathfrak{J}_r$ in such a way that*

(1) *h and k belong to the same set $\mathfrak{J}_j$ only if $n_h = n_k$ and*

(2) *the set of all U_h for $h \in \mathfrak{J}_j$ form a chain in $\mathfrak{L}$.*

Then there exists a subalgebra $\mathfrak{A}$ of $\mathfrak{B} = \mathfrak{L}^+$ for which $\mathfrak{A}^* = \mathfrak{L}$ and having $\mathfrak{A}_{hh} = \mathfrak{A}_{kk}$ whenever both h and k belong to one of the sets $\mathfrak{J}_j$. Conversely, suppose $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{A}^* = \mathfrak{L}$. Let $\mathfrak{J}_1, \ldots, \mathfrak{J}_r$ be the equivalence classes of $\mathfrak{J}$ defined by the equivalence relation $h \sim k$ if and only if $\mathfrak{A}_{hh} = \mathfrak{A}_{kk}$. Then the sets $\mathfrak{J}_j$ satisfy (1) *and* (2).

Proof. For the first part of the theorem we take for $\mathfrak{A}$ the algebra whose components are the same as those of $\mathfrak{B}$ except for the stipulated equalities $\mathfrak{A}_{hh} = \mathfrak{A}_{kk}$. Now either $\mathfrak{A}^*$ is not distributive or by repeated applications of Lemma 2 we get $\mathfrak{A}^{*+} = \mathfrak{B}$ and hence $\mathfrak{L} = \mathfrak{A}^*$.

Suppose that $\mathfrak{A}^*$ is not distributive, then it contains a prime projective root $\mathfrak{P} = [R; S, T, U; W]$ normal [**2**, §4] with respect to the chain $0 = V_0 \subset V_1 \subset \ldots \subset V_l = V$, say V_i/V_{i-1}, W/T, U/R, V_j/V_{j-1} is a sequence of transposes and $V_i \subseteq S \subseteq V_{j-1}$. (We are here using the fact that $\dim \mathfrak{A}^* = \dim \mathfrak{L}$ and so a maximal chain in $\mathfrak{L}$ is also a maximal chain in the larger lattice $\mathfrak{A}^*$.) Clearly, i and j must lie together in one of the sets $\mathfrak{J}_h$.

We refer once again to the basis chosen for V in § 5 above. For any vector v in V we speak of the first set of n_1 coefficients, ..., lth set of n_l coefficients. By our choice of basis every vector in S (or in R) has all coefficients arbitrary (but, of course, with all coefficients zero in sets $h > j$). Moreover, any vector v of W which has zero coefficients in the jth set must lie in $W \cap V_{j-1} = S$, and, similarly, if $v \in T$ or U) and has zero coefficients in the jth place then $v \in R = S \cap T$. Also, since $V_i \cap R = V_{i-1}$ any vector u of R which has zero coefficients in all sets except possibly the ith set must be zero.

Now let v be a vector in T with not all zero coefficients in the jth set. Since $U_i \subset U_j$, $A_{ij} = B_{ij} \neq 0$ and there is an n_i by n_j matrix A' such that the matrix A in $\mathfrak{A}$ having $A_{ij} = A'$ and other components zero sends v into a vector $u = Av$ having zero coefficients in all sets except the ith set and having non-zero coefficients in the ith set. This vector u cannot belong to R, but on the other hand if T is invariant under A we have $u = Av \in T \cap V_{j-1} = R$. Thus we see that T cannot be an $\mathfrak{A}$-space. This contradicition arose from our assumption that $\mathfrak{A}^*$ was not distributive. We therefore conclude that $\mathfrak{A}^*$ is distributive, and the first part of the theorem is established.

Conversely, let $\mathfrak{A}$ be a subalgebra of $\mathfrak{B}$ having $\mathfrak{A}^* = \mathfrak{L}$. Then either the theorem is true or there is a pair of indices $i < j$ for which $\mathfrak{A}_{ii} = \mathfrak{A}_{jj}$ and $U_i \not\subseteq U_j$. In this case we have from Lemma 3 that $\mathfrak{B}_{ij} = 0$ and hence that $\mathfrak{A}_{ij} = 0$.

Suppose that

$$U_{k_1}, \ldots, U_{k_s}$$

are the join-irreducible elements of $\mathfrak{L}$ properly contained in U_j. Then

$$U_j' = U_{k_1} \cup \ldots \cup U_{k_s}$$

is the unique maximal subspace of U_j. Let $R = V_{i-1} \cup U_j'$; $S = R \cup U_i$; and $T = R \cup U_j$. Since U_j does not contain U_i no U_{k_h} can contain U_i; hence

$$U_{k_h} \cap U_i \subseteq U_i' \subseteq V_{i-1}$$

(here U_i' is the unique maximal subspace of U_i). Since $\mathfrak{L}$ is distributive $R \cap U_i = U_i'$ and so S/R is a transpose of U_i/U_i'. On the other hand since $V_{i-1} \cap U_j \subseteq U_j'$ we see that $R \cap U_j = U_j'$; hence T/R is a transpose of U_j/U_j'.

By our hypothesis that $\mathfrak{A}_{ii} = \mathfrak{A}_{jj}$ we have $T - R$ operator isomorphic to $S - R$. Hence, by Lemma 1 with $W = S \cup T$ and for $a \neq 0$ in $\mathfrak{k}$, we see that the projective root $[R; S, T, Q_a; W]$ is contained in $\mathfrak{A}^*$, contrary to our hypothesis that $\mathfrak{L} = \mathfrak{A}^*$ is distributive. This contradiction arises from the assumption $U_i \not\subseteq U_j$, and so the theorem follows.

Results similar to Theorem 5 can be obtained for "super diagonal" components of $\mathfrak{A}_{ij}$ of $\mathfrak{A}$, but the theory here is not yet complete.

7. Some unsettled problems. We have seen that projective closure and the relative imbedding property are necessary conditions for closure of a lattice. It is not yet known whether these two conditions are also sufficient. The answer to this question may depend on the nature of $\mathfrak{k}$, in particular whether or not it is algebraically closed.

All of the examples known to the author of sublattices which fail to possess the relative imbedding property also fail to be projectively closed. This suggests that projective closure may imply the relative imbedding property. Again the answer may depend on the nature of $\mathfrak{k}$.

We close the present paper with an example of a lattice which does not have the relative imbedding property. Let $\mathfrak{k}$ be the rational field, let Z be a $\mathfrak{k}$-space of dimension 2, and let V be the fourfold Cartesian direct sum of Z with itself, i.e., the vectors v of V are the form (z_1, z_2, z_3, z_4) with z_i in Z. Let β be a linear transformation on Z with eigenvalues 2 and 3 and corresponding eigenvectors z' and z'' (i.e., $\beta z' = z'2$ and $\beta z'' = z''3$).

Let $\mathfrak{N}$ be the lattice of all $\mathfrak{k}$-subspaces of V and let $\mathfrak{L}$ be the finite sublattice of dimension $l = 4$ whose join-irreducible subspaces are

$$\{(z_1,0,0,0)\};\ \{(z_1,z_1,0,0)\};\ \{(z_1,\beta z_1,0,0)\};\ \{(z_1,z_2,0,0)\};$$

$$\{(0,z_1,z_2,0)\};\ \{(z_1,z_2,z_1,0)\};\ \{(0,z_1,z_2,z_3)\}.$$

(Here, for instance, $\{(z_1, z_2, z_1, 0)\}$ denotes the set of all vectors of the form $(z_1, z_2, z_1, 0)$ obtained as z_1 and z_2 range independently over Z.)

Suppose that there exists a 4-dimensional complemented modular sublattice $\mathfrak{M}$ of $\mathfrak{N}$ which contains $\mathfrak{L}$. Clearly $\mathfrak{L}$ is simple; therefore $\mathfrak{M}$ is simple. Then [1, Chap. VIII, Theorem 6] $\mathfrak{M}$ is a projective geometry of dimension 3 over a sfield $\mathfrak{K}$; since $\mathfrak{M} \subset \mathfrak{N}$ and $\mathfrak{k}$ is the rational field, $\mathfrak{K} \supseteq \mathfrak{k}$. This implies, in particular, that $\mathfrak{M}$ contains all elements of $\mathfrak{N}$ projectively related to $\mathfrak{L}$ with respect to $\mathfrak{k}$ (i.e., all Q_b with b in $\mathfrak{k}$). Then the space $\{(z, z2, 0, 0)\}$ belongs to $\mathfrak{M}$. But now $\{(z_1, z_1 2, 0, 0)\} \cap \{(z_1, \beta z_1, 0, 0)\}$ contains the non-zero vector $(z', z'2, 0, 0)$ and hence has $\mathfrak{k}$-dimension 1. Hence, dim $\mathfrak{M} > 4$.

This example, although closely related to it, does not apply to the Dilworth-Hall problem [1, p. 121, Problem 55] since as an abstract lattice $\mathfrak{L}$ can be imbedded in a complemented modular lattice of dimension 4.

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A ONE-REGULAR GRAPH OF DEGREE THREE

ROBERT FRUCHT

1. Introduction. Soon after the publication of Tutte's paper [**5**] on m-cages, H. S. M. Coxeter asked in a letter to the author whether one-regular graphs of degree 3 exist. The purpose of the following paper is to show by an example that the answer is in the affirmative.

To avoid repetitions for the definitions of the terms: finite graph, group of automorphisms of a graph, regular graphs of degree 3 (or cubical graphs), etc. the reader is referred to a former paper by the author [**4**, pp. 365-366]. Let us add only the definition of a *symmetrical* graph, as it seems that this term has not yet been defined explicitly, although it has been used by Foster [**3**], who gave a list of the symmetrical graphs known to him, and defined implicitly by Tutte [**5**], whose "s-regular cubical graphs" are nothing else than those symmetrical graphs which are connected and regular of degree 3.

Definition. A finite and connected graph is called symmetrical if for any two arcs AB and CD of the graph its group contains at least one transformation which takes the vertex A into C, and B into D.

Finally, we shall need Tutte's definition [**5**] that a graph of degree 3 is *s-regular* if it is connected, and if for any two s-arcs S_1, S_2 there is a ***unique*** transformation of the graph which carries S_1 into S_2; here an s-arc is any path $A_0 \to A_1 \to A_2 \to \dots \to A_s$ formed by s consecutive arcs $A_0 \to A_1$, $A_1 \to A_2, \dots, A_{s-1} \to A_s$ of the graph taken in a definite sense; of course $A_i \neq A_j$ for $i \neq j$.

Tutte was especially interested in the problem of finding m-cages, i.e., those s-regular graphs of degree 3 where s takes its maximal value

$$s = [\tfrac{1}{2}m + 1],$$

m being here the girth of the graph, i.e., the least number of arcs forming a closed polygon (or m-circuit); he showed also that $s < 5$ for any symmetrical graph of degree 3.

The present paper is concerned with the other extreme case, that of the lowest possible degree of symmetry a symmetrical graph of degree 3 can possess. Coxeter [**1**, p. 421] had found that there are infinitely many cubical graphs with $s = 2$, but it seems that hitherto no example with $s = 1$ was known.

In §3 such an example will be given. Unfortunately this graph has 432 vertices (hence 648 edges) so that it is practically impossible to draw it on a sheet of paper. The author hopes however that someone else will find a one-regular graph of degree 3 with fewer vertices.

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The construction of the new graph is based on a general method (described in §2) which might be of some interest in itself apart from the use made of it here, as it would allow us to find also other symmetrical graphs of degree 3.

2. A general method for constructing symmetrical graphs of degree 3. In this section H will be any group of finite order g which can be *generated by three elements of order* 2. Let a_1, a_2, a_3 be these generators, a_0 the identity, and a_4, a_5, $\ldots$, a_{g-1} the other elements of H.

Then a graph G with g vertices can be defined as follows:

(i) With every element a_i ($i = 0, 1, 2, \ldots, g-1$) of H we associate a vertex of G which shall be called a_i also, since there will be no danger of confusion.

(ii) Any two vertices a_i and a_j of G shall be joined by an arc if, and only if, in H the product $a_j a_i^{-1}$ is equal to one of the three generating elements.

In other words, the relation

$$a_j a_i^{-1} = a_k \tag{R}$$

must hold in H with $k = 1$ or 2 or 3, if a_i and a_j are the endpoints of an arc in G; and no other arcs besides those just defined are introduced.

Note that by taking the inverse of both sides, relation (R) may be given the equivalent form:

$$a_i a_j^{-1} = a_k \qquad (k = 1, 2, 3), \tag{R'}$$

since $a_k^{-1} = a_k$. It is thus seen that the relation (R) is only apparently asymmetrical in the subscripts i and j.

THEOREM 2.1. *The graph G just defined is regular of degree* 3.

(In Tutte's terminology such a graph, where each vertex is an endpoint of three arcs, is called "cubical.")

Proof. From the defining relations (R) or (R′) we have $a_j = a_k a_i$, whence it follows that for any given vertex a_i ($i = 0, 1, 2, \ldots, g-1$) of G there are just three arcs ending there, viz, those whose other endpoints are $a_1 a_i$, $a_2 a_i$, and $a_3 a_i$, respectively.

THEOREM 2.2. *The graph G is connected.*

Proof. It will be sufficient to show that for any vertex a_i of the graph G there is some s-arc joining it with a_0 (the vertex corresponding to the identity of H). Since H is generated by a_1, a_2, a_3, the element a_i of H can be written as some product of generators, say

$$a_i = a_{k_1} a_{k_2} a_{k_3} \ldots a_{k_{s-1}} a_{k_s},$$

where all the suffices k_1, k_2, $\ldots$, k_s can take only the values 1, 2, 3; moreover, it can be assumed that no "partial product"

$$a_{k_t} a_{k_{t+1}} a_{k_{t+2}} \ldots a_{k_{t+u}} \qquad (t > 1, t + u < s)$$

be equal to the identity a_0 (because otherwise it could be omitted from the total product). Then

$$a_0 \to a_{k_s},\ a_{k_s} \to a_{k_{s-1}}a_{k_s},\ a_{k_{s-1}}a_{k_s} \to a_{k_{s-2}}a_{k_{s-1}}a_{k_s},\ \ldots,$$
$$a_{k_2}a_{k_3}\ldots a_{k_s} \to a_{k_1}a_{k_2}a_{k_3}\ldots a_{k_s}$$

are arcs of the graph G since they satisfy the condition (R), and joining them in the sense indicated by the arrows we obtain the desired s-arc leading from a_0 to a_i.

THEOREM 2.3. *If a_j and a_n are any two vertices of the graph G, the group of automorphisms of G contains at least one transformation which takes a_j into a_n.*

Proof. Let T_ρ $(\rho = 0, 1, 2, \ldots, g-1)$ be that permutation of the vertices of G which takes a_i $(i = 0, 1, 2, \ldots, g-1)$ into the vertex corresponding to the product $a_i a_\rho$:

$$a_i^{T_\rho} = a_i a_\rho;$$

then T_ρ belongs to the transformations of the group of automorphisms of G. Indeed, if a_j is the other endpoint of an arc $a_i \to a_j$, we have only to show that $a_i^{T_\rho} \to a_j^{T_\rho}$ is also an arc of G. But by (R), $a_j a_i^{-1} = a_k$ (where $k = 1, 2$, or 3); hence

$$a_j^{T_\rho} \cdot (a_i^{T_\rho})^{-1} = (a_j a_\rho)(a_i a_\rho)^{-1} = a_j a_\rho a_\rho^{-1} a_i^{-1} = a_j a_i^{-1} = a_k,$$

and this is, by (R), just the condition for $a_i^{T_\rho}$ and $a_j^{T_\rho}$ to be endpoints of an arc in G.

Now, if a_j and a_n are the two vertices considered in the theorem, the transformation T_j takes any a_i into $a_i a_j$, and hence a_0 into $a_0 a_j = a_j$. In an analogous manner T_n takes a_0 into a_n. Hence the product $T_j^{-1}T_n$, i.e., the inverse transformation T_j^{-1}, followed by T_n, takes a_j into a_n, and thus satisfies Theorem 2.3.

It might be remarked as a corollary that the g transformations $T_0, T_1, \ldots, T_{g-1}$ constitute a group simply isomorphic to H.

The theorem thus proved tells us, in other words, that the group of automorphisms of G is transitive on the vertices; note that this is less than symmetry (as defined in §1) which requires transitivity at least on the 1-arcs. In order to obtain symmetrical graphs of degree 3 it will be necessary to impose a further condition on the group H. (So far it had only been required that H be generated by three elements of order 2.) Since this condition has to do with the automorphisms of the group H, it will be convenient to state first the following theorem:

THEOREM 2.4. *Let the group H (generated by three elements of order 2) admit an automorphism ϕ which permutes the three generators of H. If this automorphism carries any element a_i of H into a_i^ϕ, then the corresponding permutation of the vertices of G is a transformation of G belonging to its group of automorphisms.*

Proof. We have only to show that, if a_i and a_j are endpoints of an arc in G, so also are a_i^ϕ and a_j^ϕ. Since ϕ is an automorphism of H,

$$a_j^\phi \,.\, (a_i^\phi)^{-1} = a_j^\phi \,.\, (a_i^{-1})^\phi = (a_j a_i^{-1})^\phi,$$

and replacing $a_j a_i^{-1}$ by a_k (where $k = 1, 2$, or 3), we have

$$a_j^\phi \,.\, (a_i^\phi)^{-1} = a_k^\phi.$$

But we made the assumption that ϕ only permutes the generators a_1, a_2, a_3; hence a_k^ϕ is also one of the generating elements, say a_q ($q = 1, 2, 3$). Thus we have obtained

$$a_j^\phi \,.\, (a_i^\phi)^{-1} = a_q \qquad (q = 1, 2, 3);$$

and this is, according to (R), the condition for a_i^ϕ and a_j^ϕ to be endpoints of an arc in G.

Having proved Theorem 2.4, we now give a sufficient condition for obtaining symmetrical graphs of degree 3.

THEOREM 2.5. *If the group H admits an automorphism such that the three generators of order 2 undergo a* cyclic *permutation, then the graph G is symmetrical.*

Proof. Let $a_i \to a_j$ and $a_q \to a_r$ be any two arcs in G; then Theorem 2.5 will be proved if we can show that the group of automorphisms of G contains at least one transformation θ fulfilling the two conditions:

$$\text{(C)} \qquad a_i^\theta = a_q \text{ and } a_j^\theta = a_r.$$

To obtain such a transformation θ we proceed as follows: as in the proof of Theorem 2.3, let T_p be that transformation of G which replaces any vertex a_s by $a_s a_p$, let ϕ be the transformation of G corresponding (by Theorem 2.4) to the automorphism ϕ of H whose existence is supposed in Theorem 2.5; then also the three products (read from left to right)

$$\theta_n = T_i^{-1}\phi^n T_q \qquad (n = 1, 2, 3)$$

are transformations belonging to the group of the graph. We will show that just one of them fulfils the two conditions (C).

It is obvious that the first of these conditions is satisfied for each value of n; indeed, T_i^{-1} replaces a_i by a_0, any power of ϕ leaves a_0 fixed, and T_q takes a_0 into a_q. As to the second condition, the following argument may be used:

By (R) we have not only $a_j = a_k a_i$ ($k = 1, 2, 3$), but also that $a_r a_q^{-1}$ is a generator,

$$a_r a_q^{-1} = a_t \qquad (t = 1, 2, \text{or } 3).$$

Now, since ϕ is supposed to produce a cyclic permutation on the generators a_1, a_2, a_3, some power of ϕ will replace a_k by a_t. Let ϕ^n be that power of ϕ, then

$$a_k^{\phi^n} = a_t \qquad (n = 1, 2, \text{ or } 3),$$

and it follows readily that also the second condition (C) is satisfied by the transformation $\theta = \theta_n = T_i^{-1}\phi^n T_q$, since

$$a_j^{\theta_n} = (a_k a_i)^{T_i^{-1}\phi^n T_q} = a_k^{\phi^n T_q} = a_t^{T_q} = (a_r a_q^{-1})^{T_q} = a_r.$$

We close this section by giving three examples of symmetrical graphs of degree 3 which can be obtained by Theorem 2.5.

(1) The simplest example of a group H which can be generated by three (but not less than three) elements of order 2 (and which admits an automorphism of the kind established in Theorem 2.5) is the direct product $\{a_1\} \times \{a_2\} \times \{a_3\}$ of order 8. It is easy to see that the corresponding graph G is that of the vertices and edges of a cube.

(2) The symmetric group of degree 4 and order $4! = 24$ can be generated by $a_1 = (1\ 2)$, $a_2 = (1\ 3)$, $a_3 = (1\ 4)$. The corresponding graph turns out to be Foster's III-13 [**3**, Fig. 9]; it is called $\{12\} + \{12/5\}$ by Coxeter [**1**, pp. 439, 440].

(3) An apparently new symmetrical graph with 64 vertices and girth 8 corresponds to the group of order 64 which can be generated by the following permutations on twelve symbols:

$$\begin{aligned} a_1 &= (1\ 2)(3\ \ 4)(\ 9\ 11), \\ a_2 &= (1\ 3)(5\ \ 6)(\ 7\ \ 8), \\ a_3 &= (5\ 7)(9\ 10)(11\ 12). \end{aligned}$$

(As an abstract group, it can be characterized by the condition that all the commutators be invariant elements.)

3. Example of a one-regular graph of degree 3. The symmetrical graphs mentioned as examples at the end of the foregoing section are 2-regular. It is easy to see that this is due to the fact that in these examples the group H does not only admit an automorphism allowing a cyclic permutation of the generators, but is formally symmetrical in the three generators, admitting, e.g., an automorphism which leaves a_3 fixed and replaces a_1 by a_2.

This fact seemed to justify the author's hope of obtaining a one-regular graph of degree 3 by the method of Theorem 2.5, if a group H could be found which admits automorphisms producing cyclic permutations of the three generators of order 2, but no automorphism leaving a_3 fixed while interchanging a_1 and a_2. We will show that such a group is that generated by the following three permutations on nine symbols:

$$\begin{aligned} a_1 &= (1\ 2)(3\ 5)(4\ 8), \\ a_2 &= (1\ 3)(2\ 6)(5\ 9), \\ a_3 &= (1\ 4)(2\ 3)(6\ 7). \end{aligned}$$

In this section the letter H will be used to denote the group with these generators.

THEOREM 3.1. *The group H just defined admits an automorphism ϕ satisfying the conditions:*

$$a_1^{\phi} = a_2, \quad a_2^{\phi} = a_3, \quad a_3^{\phi} = a_1;$$

but there is no automorphism ψ of H satisfying the conditions:

$$a_1^{\psi} = a_2, \quad a_2^{\psi} = a_1, \quad a_3^{\psi} = a_3.$$

Proof. If b is the following permutation:

$$b = (1\ 3\ 2)(4\ 5\ 6)(7\ 8\ 9),$$

it is immediately seen that

$$b^{-1}a_1b = a_2, \quad b^{-1}a_2b = a_3, \quad b^{-1}a_3b = a_1;$$

whence it follows that the automorphism

$$a_i^{\phi} = b^{-1}a_ib \qquad (i = 0, 1, 2, \ldots, g - 1)$$

satisfies the conditions of Theorem 3.1. It may be remarked that this is an inner automorphism of the group H, since b is an element of H; indeed a somewhat lengthy computation shows that

$$b = \{(a_1a_3)^2(a_1a_2a_1a_3)^2\}^3 \cdot (a_1a_2a_1a_3)^3a_1a_3.$$

To prove that no automorphism ψ of H can exist which leaves a_3 fixed while interchanging a_1 and a_2, compute the product

$$e = a_3a_2(a_3a_1)^2(a_3a_2)^2a_1a_2,$$

from which it is found that $e = a_0$. Assuming the existence of an automorphism ψ and applying it to the product e just introduced, we would have

$$e^{\psi} = a_3a_1(a_3a_2)^2(a_3a_1)^2a_2a_1;$$

but since $e = a_0$, the left-hand side of the last equation must likewise be the identity a_0; the right-hand side however is found equal to

$$(1\ 6\ 8)(2\ 5\ 7)(3\ 4\ 9),$$

hence distinct from the identity. This contradiction shows the impossibility of the existence of ψ.

THEOREM 3.2. *The graph G of degree 3 which corresponds to the group H is one-regular.*

Proof. That G is at least one-regular follows from Theorems 2.5 and 3.1. It remains to be shown that G is not 2-regular. This can be done as follows:

We have already seen, in the proof of Theorem 2.2, that the endpoints of the s-arcs beginning with a_0 are the "non-trivial" products of s generating elements (where "non-trivial" means that there are no partial products equal to the identity); e.g., the six 2-arcs beginning with a_0 are

$$a_0 \to a_1 \to a_2a_1, \quad a_0 \to a_1 \to a_3a_1, \quad a_0 \to a_2 \to a_1a_2,$$

$$a_0 \to a_2 \to a_3a_2, \quad a_0 \to a_3 \to a_1a_3, \quad a_0 \to a_3 \to a_2a_3.$$

In an analogous manner we can form the twelve 3-arcs beginning with a_0, etc. By computing all the products of 2, 3, 4, . . . generators of our group H, it turns out that these endpoints of the s-arcs beginning with a_0 are all different *so long as* $s \leqslant 5$. But of the endpoints of the 96 six-arcs beginning with a_0 only 77 are distinct, since it turns out that there are fifteen pairs of such 6-arcs with a common endpoint, and two triples of such 6-arcs, viz, those ending with

$$a_1a_3a_1a_2a_1a_2 = a_2a_1a_2a_3a_2a_3 = a_3a_2a_3a_1a_3a_1 = (1\ 5\ 9)(2\ 4\ 8)(3\ 6\ 7)$$

or

$$a_1a_3a_1a_3a_2a_3 = a_2a_1a_2a_1a_3a_1 = a_3a_2a_3a_2a_1a_2 = (1\ 9\ 5)(2\ 8\ 4)(3\ 7\ 6).$$

These two vertices are thus characterized as the only endpoints of ***triples*** of 6-arcs beginning with a_0; therefore no transformation of G can exist that leaves a_0 fixed but carries one of these 6-arcs into a 6-arc ending at some other vertex. Now the two triples of 6-arcs just considered begin with one of the following three 2-arcs:

$$a_0 \to a_2 \to a_1a_2, \quad a_0 \to a_3 \to a_2a_3, \quad \text{or} \quad a_0 \to a_1 \to a_3a_1;$$

hence no transformation of G can take one of these three 2-arcs into any of the other three 2-arcs beginning with a_0. This means that G is not 2-regular.

COROLLARY. *The graph G is of girth* 12.

Indeed a 12-circuit can be formed with two of the 6-arcs of one of the triples just mentioned, but no m-circuit exists with $m < 12$.

THEOREM 3.3. *The one-regular graph just found has* 432 *vertices.*

Proof. Since the number of vertices of G is equal to the order of the group H, we have only to prove that the group generated by

$$a_1 = (1\ 2)(3\ 5)(4\ 8), \quad a_2 = (1\ 3)(2\ 6)(5\ 9), \quad a_3 = (1\ 4)(2\ 3)(6\ 7),$$

is of order 432.

It is easy to see that the order of this group H is either 432 *or some factor contained in* 432. Indeed, the generators a_1, a_2, a_3, and hence any permutation of H, leave fixed the following triple system of 9 elements:

129; 137; 145; 168; 238; 246; 257; 349; 356; 478; 589; 679;

and it has been pointed out by Emch [2] that the group of all the permutations which leave a triple system of nine elements invariant, has the order 432, since it is simply isomorphic to the holomorph of the noncyclic group of order 9.

Hence our group H is simply isomorphic either to the same holomorph of order 432 *or to some proper subgroup of it.* It remains only to be shown that this second alternative does not take place, i.e., that the order of H cannot be less than 432.

Our group H is transitive, since the eight permutations $a_1, a_2, a_3,$

$$a_1a_3a_1 = (1\ 5)(2\ 8)(6\ 7), \quad a_1a_2a_1 = (1\ 6)(2\ 5)(3\ 9),$$

$$a_3a_1a_3 = (1\ 8)(2\ 5)(3\ 4), \quad a_2a_1a_2 = (1\ 9)(3\ 6)(4\ 8),$$

$$a_2a_3a_2a_3 = (1\ 7\ 3)(2\ 6\ 4),$$

replace the symbol "1" respectively by 2, 3, . . . , 9. According to a well-known theorem, the order of H will be nine times that of the subgroup H_1 formed by all the permutations of H each of which leaves the symbol "1" fixed. But this subgroup H_1 contains the following three permutations:

$$a_3a_2a_3 = (24)(37)(59), \quad a_3a_2a_1a_3a_1a_3 = (2\ 5\ 9\ 4)(3\ 6\ 7\ 8),$$

$$(a_1a_3)^4 \cdot a_3a_2a_1a_3a_1a_2 = (2\ 7\ 8\ 9\ 3\ 6)(4\ 5).$$

It is readily shown by computing all the products of the last two permutations (which are even) that it is possible to form 24 different even permutations which leave the symbol "1" fixed. Hence H_1 (containing also the odd permutation $a_3a_2a_3$) cannot have an order less than 48, and H cannot be of order less than 432.

COROLLARY. *The group of automorphisms of the graph described in this section is of order* 1296.

Indeed it is a regular permutation group on the 1296 one-arcs.

Addendum.

It was kindly pointed out by Coxeter (in a letter to the author) that the one-regular graph described in the last section could be derived from the regular hyperbolic tesselation $\{12, 3\}$ by making appropriate identifications; in other words, it could be embedded in a surface of characteristic -108 (or genus 55) so as to form a map of 108 dodecagons (in agreement with the Corollary to Theorem 3.2).

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F-EQUATION FOURIER TRANSFORMS

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Introduction. For solutions of the differential difference 'F-equation'

$$\frac{d}{dz}F(z,\alpha) = F(z,\alpha + 1), \tag{1}$$

there are representations as a generating series [**5**],

$$F(z,\alpha) = \sum_{n=0}^{\infty} f(\alpha + n)\frac{z^n}{n!}, \tag{2}$$

and as a contour integral [**3**],

$$F(z,\alpha) = \int_C \exp(\alpha s + ze^s)g(s)ds. \tag{3}$$

Integrals of the type (3) have certain formal properties which are reflected in identities satisfied by solutions of (1). In this note are given some relations which hold when the integral is taken to be a complex Fourier transform with respect to the variable α. The special properties of Fourier integrals lead to some additional results in this case.

The interest of the equation (1) lies in the fact, which was pointed out by Truesdell [**5**], that many useful special functions can be expressed as solutions of the equation. Working with the F-equation, Truesdell was able to classify many of the formulae satisfied by such functions, to generalize some of them, and to discover additional relations.

1. Fourier integral representations. If a solution $F(z,\alpha)$ of (1) is represented in the forms (2) and (3), we may call $f(\alpha)$ the coefficient function, and $g(s)$ the spectrum function, of the given solution. For the case in hand, these two functions are a Fourier pair. That is, the formulae

$$f(\alpha) = \int_{-\infty}^{\infty} e^{2\pi i\alpha s}g(s)ds, \quad g(s) = \int_{-\infty}^{\infty} e^{-2\pi i\alpha s}f(\alpha)d\alpha, \tag{4}$$

hold in some sense. The factor 2π has been written in the exponentials for convenience. We consider, therefore, solutions with the representations

$$F(z,\alpha) = \int_{-\infty}^{\infty} \exp(2\pi i\alpha s + ze^{2\pi is})g(s)ds = \sum_{n=0}^{\infty} f(\alpha + n)\frac{z^n}{n!}. \tag{5}$$

Since the relation between $f(\alpha)$ and $g(s)$ is skew reciprocal, we may, with the alteration of a sign, form a second solution in which their roles are interchanged. In this way a conjugate solution

$$G(z,\alpha) = \int_{-\infty}^{\infty} \exp(-2\pi i\alpha s + ze^{-2\pi is})f(s)ds = \sum_{n=0}^{\infty} g(\alpha + n)\frac{z^n}{n!} \tag{6}$$

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is constructed, whose coefficient function is the spectrum function of $F(z, \alpha)$. The relation between $F(z, \alpha)$ and $G(z, \alpha)$ is skew reciprocal with respect to α.

Example 1. Let

$$f(\alpha) = \begin{cases} \dfrac{1}{\Gamma(\mu)}\alpha^{\mu-1}, & \alpha > 0 \\ 0 & , \alpha < 0 \end{cases} \qquad 0 < R(\mu) < 1.$$

Then [2, § 521],

$$g(s) = (2\pi i s)^{-\mu} e^{-\pi i \mu},$$

and we have

$$F(z,\alpha) = \frac{1}{\Gamma(\mu)} \sum_{n=0}^{\infty} (\alpha + n)^{\mu-1} \frac{z^n}{n!},$$

$$G(z,\alpha) = (2\pi i)^{-\mu} e^{-\pi i \mu} \sum_{n=0}^{\infty} \frac{z^n}{(\alpha + n)^{\mu} n!}.$$

These functions are related to the generalized zeta function and to Spence's transcedent [5, p. 172].

THEOREM I. *The functions*

$$\exp(te^{-2\pi i\alpha})F(z,\alpha),\ \exp(ze^{2\pi i\alpha})G(t,\alpha), \tag{7}$$

are solutions of (1) *in the variables* z, α *and* t, α, *respectively. For all* z, t, *they are a Fourier pair in* α.

Proof. The exponential factors are periodic in α of period 1, which clearly justifies the first statement. To demonstrate the transform property, we note that the transform of $g(s + n)$ is $f(\alpha)e^{-2\pi i n\alpha}$, so that if n is an integer

$$\int_{-\infty}^{\infty} \exp(2\pi i\alpha s + ze^{2\pi i s})g(s + n)ds = \int_{-\infty}^{\infty} \exp(2\pi i\alpha(t - n) + ze^{2\pi i t})g(t)dt$$
$$= e^{-2\pi i\alpha n}F(z,\alpha).$$

Multiply by $t^n/n!$ and sum. Using (2) and (3) we find that

$$\int_{-\infty}^{\infty} \exp(2\pi i\alpha s + ze^{2\pi i s})G(t,s)ds = \exp(te^{-2\pi i\alpha})F(z,\alpha), \tag{8}$$

which expresses the result.

We may therefore introduce parameters into our formulae by replacing any pair $f(\alpha)$, $g(s)$ by the pair (7). We shall see below that any valid formula containing the pair $f(\alpha)$, $g(s)$ remains valid if this substitution is made.

We investigate the sense in which the above formulae hold. Since we shall have $f(\alpha)$ and its transform $g(s)$ tending ultimately to zero except in sets of small measure, it is clear that the series in (5) and (6) converge for all z like the exponential function, and define entire analytic functions of z, for almost all α. Since s is real, we have

$$|\exp(ze^{2\pi i s})| < e^{|z|}. \tag{9}$$

Hence, if we suppose that f and g are $L(-\infty, \infty)$, or $L^2(-\infty, \infty)$, the integrals (5) and (6) converge, or converge in mean, uniformly with respect to z. Thus, if (4) hold in either sense, then (5), (6), and (7) hold in the same sense for all z.

Let $f(\alpha)$ be $L^p(-\infty, \infty)$, then except in a set of arbitrarily small measure, the members of the sequence $|f(\alpha+n)|$ $(n = 0, 1, 2, \ldots)$ are less than some function $M(\alpha)$. Hence, except in this set, the series (5) converges like the exponential function and we have (see also [**5**, Corollary 11.6]).

THEOREM II. *If $f(\alpha)$ is $L^p(-\infty, \infty)$, $p > 0$, the function $F(z, \alpha)$ given by* (5) *satisfies*

$$|F(z,\alpha)| < M(\alpha)e^{|z|}, \tag{10}$$

where $M(\alpha)$ is defined for almost all α.

2. Product formulae. The results of this section concern solutions of (1) whose coefficient or spectrum functions are products of the corresponding functions for known solutions. We use the formulae for Fourier transforms given in [**4**, Chap. 2].

THEOREM III. *Let $F_1(z, \alpha)$ and $F_2(z, \alpha)$ have coefficient functions $f_1(\alpha)$, $f_2(\alpha)$, and spectrum functions $g_1(s)$, $g_2(s)$, respectively. Let $g_1(s)$ and $g_2(s)$ be $L(-\infty, \infty)$. Then the unique solution $F(z, \alpha)$ of* (1) *with coefficient function $f_1(\alpha)f_2(\alpha)$ is given by*

$$\begin{aligned} F(z,\alpha) &= F_1^* F_2(z,\alpha) \\ &= \int_{-\infty}^{\infty} e^{2\pi i\alpha s} g_1(s) F_2(ze^{2\pi is},\alpha)\,ds \\ &= \int_{-\infty}^{\infty} e^{2\pi i\alpha s} g_2(s) F_1(ze^{2\pi is},\alpha)\,ds \\ &= \int_{-\infty}^{\infty} \exp(2\pi i\alpha s + ze^{2\pi is}) g(s)\,ds \\ &= \sum_{n=0}^{\infty} f_1(\alpha+n) f_2(\alpha+n)\frac{z^n}{n!}, \end{aligned} \tag{11}$$

where

$$g(s) = \int_{-\infty}^{\infty} g_1(s-t) g_2(t)\,dt \tag{12}$$

is the transform of the product $f_1(\alpha)f_2(\alpha)$.

Proof. Under these hypotheses, the first two integrals converge absolutely, since the exponential factors are bounded and the F factors bounded and periodic in s. The convergence is uniform with respect to z and α, provided that the absolute values of these variables are bounded. The integrals define, therefore, analytic functions of z. Since [**4**, Theorem 41] $g(s)$ is $L(-\infty, \infty)$, the third integral also converges absolutely and uniformly. Each expression is a solution of (1), as is easily verified. Finally, the coefficient function of each solution is

$f_1(a)f_2(a)$, since each expression reduces to this value when $z = 0$. It is clear that the coefficient function determines the solution uniquely, and the theorem follows.

This theorem is an adaptation to the present case of a result given in [3].

Example. The product solution of the pair given in Example 1 is

$$F^*G(z,a) = \frac{(2\pi i)^{-\mu}e^{-\pi i\mu}}{\Gamma(\mu)}\sum_{n=0}^{\infty}\frac{z^n}{(a+n)n!}$$

$$= \frac{(2\pi i)^{-\mu}e^{-\pi i\mu}}{\Gamma(\mu)a}{}_1F_1(a;\,a+1;\,z),$$

where ${}_1F_1$ is the confluent hypergeometric function.

THEOREM IV. *Let $f_1(a)$, $f_2(a)$ and their transforms be $L^2(-\infty,\infty)$. Then*

$$\int_{-\infty}^{\infty}F_1^*G_2(z,a)da = \int_{-\infty}^{\infty}F_2^*G_1(z,a)da = Ce^z. \tag{13}$$

Proof. Substitute the series expansion for the solution on the left. We find

$$\int_{-\infty}^{\infty}\sum_{n=0}^{\infty}f_1(a+n)g_2(a+n)\frac{z^n}{n!}da = \sum_{n=0}^{\infty}\frac{z^n}{n!}\int_{-\infty}^{\infty}f_1(a+n)g_2(a+n)da$$

$$= e^z\int_{-\infty}^{\infty}f_1(a)g_2(a)da = Ce^z,$$

the last step holding since the integral is independent of n. In view of the Parseval theorem [4, p. 69], the integral corresponding to the second expression has the same value C. This proves (13).

The next result is a sort of dual to Theorem III, since it involves solutions whose spectrum functions are products. The conditions are somewhat different however.

THEOREM V. *Let $f_1(a)$ and $f_2(a)$ be $L^2(-\infty,\infty)$, and let their convolution be denoted by $f(a)$. Then the solution of* (1) *with coefficient function $f(a)$ is given by*

$$H(z,a) = F_1^0F_2(z,a)$$

$$= \int_{-\infty}^{\infty}F_1(z,a-s)f_2(s)ds$$

$$= \int_{-\infty}^{\infty}F_2(z,a-s)f_1(s)ds \tag{14}$$

$$= \int_{-\infty}^{\infty}\exp(2\pi ias + ze^{2\pi is})g_1(s)g_2(s)ds$$

$$= \sum_{n=0}^{\infty}f(a+n)\frac{z^n}{n!}.$$

Proof. The conditions stated ensure that $f(a)$ exists, that $g_1(s)$ and $g_2(s)$ are

$L^2(-\infty,\infty)$, and therefore that $g_1(s)\, g_2(s)$ is $L(-\infty,\infty)$. The third integral is therefore absolutely and uniformly convergent. Since F_1 and F_2 are also $L^2(-\infty,\infty)$, uniformly with respect to z and a, in their dependence upon s, the first two integrals also converge absolutely and uniformly. The four expressions therefore define analytic functions of z which are entire. As in the proof of Theorem III, it is clear that each expression is a solution of (1) and that each reduces to the same coefficient function $f(a)$ when $z = 0$. This completes the proof.

THEOREM VI. *$H(z, a)$ being defined as in Theorem V, we have*

$$\int_{-\infty}^{\infty} F_1(x,a - s)F_2(y,\beta + s)ds = H(x + y, a + \beta). \tag{15}$$

Proof. From (5) it is clear that $F_1(x, a + s)$ is the transform of $\exp(2\pi ias+xe^{2\pi it})g_1(s)$ and $F_2(y,\beta+s)$ the transform of $\exp(2\pi i\beta s+ye^{2\pi it})g_2(s)$. The integral (15) is the convolution of these two solutions and therefore its transform is the product of their transforms. But the third integral of (14) shows that this transform is precisely $H(x + y, a + \beta)$.

THEOREM VII. *Let $F_i(z, a)$ and $G_i(z, a)$ be pairs of conjugate solutions for $i = 1, 2$. Then*

$$F_1^0F_2(z,a), \quad G_1^*G_2(z,a) \tag{16}$$

are a pair of conjugate solutions.

Proof. The coefficient functions of the pair (16) are the Fourier pair

$$\int_{-\infty}^{\infty} f_1(a - s)f_2(s)ds, \quad g_1(a)g_2(a).$$

We shall now give some examples of these formulae. If $f(a)$ is a rational function of a, expressible by quotients of gamma functions, the series (5) takes the form of a generalized hypergeometric function. The transform $g(s)$ is a sum of rational functions multiplied by exponentials, and the conjugate series (6) can also be expressed in terms of hypergeometric series in this case.

Example 2. [**2**, § 438]. Let

$$f_\beta(a) = \begin{cases} e^{-\beta a}, & a > 0, \\ 0\ , & a < 0. \end{cases}$$

Then

$$g_\beta(s) = (\beta - 2\pi is)^{-1}, \qquad R(\beta) > 0,$$

$$F_\beta(z,a) = \exp(-\beta a + ze^{-\beta}), \qquad a > 0,$$

$$G_\beta(z,a) = (\beta - 2\pi ia)^{-1}{}_1F_1(a - \beta/2\pi i; a + 1 - \beta/2\pi i; z).$$

We have

$$F_\beta^* F_\gamma = F_{\beta+\gamma}(z,a), \qquad R(\beta) > 0, R(\gamma) > 0,$$

$$G_\beta^* G_\gamma(z,a) = (\beta - 2\pi i a)^{-1}(\gamma - 2\pi i a)^{-1}$$
$$\times\ {}_2F_2(a - \beta/2\pi i, a - \gamma/2\pi i; a + 1 - \beta/2\pi i; a + 1 - \gamma/2\pi i; z),$$
$$F_\beta^0 F_\gamma = (\gamma - \beta)^{-1}[\exp(-\beta a + ze^{-\beta}) - \exp(-\gamma a + ze^{-\gamma})],$$

using [**2**, § 448], and

$$G_\beta^0 G_\gamma = G_{\beta+\gamma}.$$

From Theorem IV we have [**2**, p. 30]

$$\int_{-\infty}^{\infty} F_\beta^* G_\gamma(z,a)\,da = \frac{\exp(z - \beta\gamma/2\pi i)}{2\pi i}\mathrm{Ei}\left(\frac{\beta\gamma}{2\pi i}\right).$$

Example 3. Let [**2**, § 708],

$$f_\beta(a) = \exp(-\pi a^2/\beta), \qquad R(\beta) > 0,$$

$$g_\beta(s) = \sqrt{\beta}\exp(-\pi\beta s^2) = \sqrt{\beta}\, f_{1/\beta}(s).$$

We then have

$$F_\beta(z,a) = \sum_{n=0}^{\infty} \exp(-\pi(a+n)^2/\beta)\frac{z^n}{n!},$$
$$G_\beta(z,a) = \sqrt{\beta} F_{1/\beta}(z,a).$$

By Theorem I, the functions

$$\sum_{n=0}^{\infty} \exp(-\pi(a+n)^2/\beta + te^{-2\pi ia})\frac{z^n}{n!}, \qquad \sqrt{\beta}\sum_{n=0}^{\infty} \exp(-\pi\beta(a+n)^2 + ze^{2\pi ia})\frac{t^n}{n!}$$

are a Fourier pair. Also

$$F_\beta^* F_\gamma = F_{\beta+\gamma}; \quad F_\beta^0 F_\gamma = \sqrt{\beta+\gamma}\,F_{\beta\gamma/\beta+\gamma}.$$

By Theorem IV,

$$\int_{-\infty}^{-\infty} F_\beta^* F_\gamma(z,a)\,da = \sqrt{\beta\gamma/(1+\beta\gamma)}\,e^{z},$$

and by Theorem VI,

$$\int_{-\infty}^{\infty} F_\beta(x, a+s)F_\gamma(y, b-s)\,ds = \sqrt{\beta+\gamma}\,F_{\beta\gamma/(\beta+\gamma)}(x+y, a+b).$$

3. Generating series. In this section we shall use the Poisson formula for Fourier transforms to evaluate certain bilateral generating series for conjugate solutions. The Poisson formula [**1**, p. 33]

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} g(n), \tag{17}$$

holds if, for instance, $f(a)$ is $L(-\infty, \infty)$, of bounded variation, and tends monotonically to zero at infinity.

THEOREM VIII. *Let $F(z, \alpha)$ and $G(z, \alpha)$ be a pair of conjugate solutions of* (1). *Let the Fourier pair determining them be such that* (17) *holds. Then*

$$\sum_{n=-\infty}^{\infty} F(z,\alpha + n)e^{2\pi in\theta} = \exp(ze^{-2\pi i\theta})K(\theta,\alpha), \tag{18}$$

$$\sum_{n=-\infty}^{\infty} G(z,\alpha + n)e^{2\pi in\theta} = \exp(ze^{-2\pi i\theta})L(\theta,\alpha),$$

where $K(\theta, \alpha)$ and $L(\theta, \alpha)$ are periodic of period 1 *in α. Furthermore,*

$$L(\theta,\alpha) = e^{-2\pi i\alpha\theta}K(-\alpha,\theta). \tag{19}$$

Proof. We assume that θ and α are real, though the result may hold more generally. Under the conditions stated, and in view of Theorem II, the generating series (18) define analytic functions of z regular for $z = 0$. Let $S = S(z, \alpha, \theta)$ denote the first of these series. Then

$$\frac{dS}{dz} = \sum_{-\infty}^{\infty} F(z,\alpha + n + 1)e^{2\pi in\theta} = e^{-2\pi i\theta}S,$$

using (1) and changing the summation index from n to $n + 1$. Integrating this equation, we find that S is of the form (18). The periodicity of K and L is evident. Setting $z = 0$, we have to prove (19). Now the transform of $f(s + a)$ is $g(s)e^{2\pi isa}$, and the transform of $g(s + a)$ is $f(s)e^{-2\pi isa}$. Inserting these in (17) and taking account of (18), we find precisely (19). The formula taking account of the translations holds under the same conditions as (17). This proves the theorem. Two particular cases of this result may be mentioned.

(a) If $F(z, \alpha)$ and $G(z, \alpha)$ are conjugate and (17) holds, then

$$\sum_{n=-\infty}^{\infty} F(z,n) = \sum_{n=-\infty}^{\infty} G(z,n) = Ke^{z}. \tag{20}$$

Example 4. Taking $f_\beta(\alpha) = \exp(-\pi\alpha^2/\beta)$ as in Example 3, we find that

$$\beta^{-\frac{1}{2}}\sum_{n=-\infty}^{\infty} F_\beta(z,n) = \sum_{n=-\infty}^{\infty} F_{1/\beta}(z,n) = \mathfrak{D}(\beta)e^{z},$$

where [1, p. 152]

$$\mathfrak{D}(x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x}$$

is related to the theta functions.

(b) If $F(z, \alpha)$ is any solution of (1) such that the indicated series converge,

$$\sum_{n=-\infty}^{\infty} F(z,\alpha + n)e^{2\pi in\theta} = \exp(ze^{-2\pi i\theta})\sum_{n=-\infty}^{\infty} f(\alpha + n)e^{2\pi in\theta}. \tag{21}$$

Example 5 [2, § 444].

$$f(\alpha) = (\alpha^2 + b^2)^{-1}; \quad g(s) = 2\pi b^{-1}e^{-2\pi b|s|}.$$

Then

$$F(z,a) = (a^2 + b^2)^{-1}{}_2F_2(a + ib, a - ib; a + ib + 1, a - ib + 1; z),$$

$$G(z,a) = 2\pi b^{-1}\exp(-2\pi ab + ze^{-2\pi b}), \qquad a > 0.$$

To evaluate the sum $K(\theta, a)$ on the right-hand side of (21), we use (17), since the sum over $g(s)$ is more tractable. Letting $[x]$ denote the greatest integer contained in x, we find

$$K(\theta,a) = \frac{e^{-2\pi ia\theta}}{4\pi b}\left\{\frac{\exp(-2\pi b\{\theta + 1 - [\theta]\} + 2\pi i[\theta + 1]a)}{1 - \exp(-2\pi b - 2\pi ia)}\right.$$
$$\left. + \frac{\exp(-2\pi b\{\theta - [\theta]\} + 2\pi i\theta a)}{1 - \exp(-2\pi b + 2\pi ia)}\right\}.$$

As another example of some of the formulae of this paper, we include

Example 6. For $n = 1, 2, \ldots,$ let [4, p. 76]

$$f_n(a) = \frac{1}{(2^{n-1}n!)^{\frac{1}{2}}}\, e^{-\pi a^2}H_n(2\sqrt{\pi}a), \quad g_n(s) = i^{-n}f_n(s).$$

Here $H_n(x)$ is the nth Hermite polynomial. The $f_n(a)$ form a complete orthonormal set in $L^2(-\infty, \infty)$, hence the solutions $F_n(z, a)$ form a base for the solutions of (1) whose coefficient functions are L^2. We have

$$G_n(z,a) = i^{-n}F_n(z,a),$$

$$F_n^*F_m(z,a) = \sum_{n=0}^{\infty} a(m,n,p)F_p(z,a),$$

$$F_n^0F_m(z,a) = i^{m+n}\sum_{p=0}^{\infty} i^{-p}a(m,n,p)F_p(z,a),$$

where

$$a(m,n,p) = \int_{-\infty}^{\infty} f_m(s)f_n(s)f_p(s)ds$$

is symmetric in m, n, and p, and vanishes if $m + n + p$ is odd. Some further expansions in terms of these coefficients are

$$\int_{-\infty}^{\infty} e^{2\pi ias}F_m^*F_p(z,s)ds = \exp(ze^{-2\pi ia})\sum_{r=0}^{\infty} i^r a(m,p,r)f_r(a),$$

$$\int_{-\infty}^{\infty} e^{2\pi ias}F_m^0F_p(z,s)ds = \exp(ze^{-2\pi ia})i^{m+p}\sum_{r=0}^{\infty} a(m,p,r)f_r(a),$$

and

$$\int_{-\infty}^{\infty} F_m(x,a + s)F_p(y,\beta - s)ds = i^{m+p}\sum_{r=0}^{\infty} i^r a(m,p,r)F_r(x + y,a + \beta).$$

Also (18) hold for the pair $F_m(z, a)$, $G_m(z, a)$, with

$$K_m(\theta,a) = i^m L_m(\theta,a) = i^m e^{-2\pi ia\theta} K_m(-a,\theta).$$

The function $K_0(\theta, p)$, p an integer, is a theta function.

Although solutions $F(z, a)$ of (1) of the Fourier type possess convenient formal properties, their behaviour as functions of z is rather restricted, as Theorem II shows. However, many of these properties carry over to integrals of Laplace transform type which represent many of the functions to which the equation (1) may be applied. It may be noted that the Fourier solutions exhibit markedly different behaviour as functions of z and of a, since they are analytic in z but may only be measurable as functions of a.

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